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A GENERAL COMPUTER PROGRAM FOR IONOSPHERIC RAY-TRACING

by

VICTOR H. GONZALES

NsG-24-59

1 August 1961

Sponsored by:

National Aeronautics and Space Administration
Washington 25, D. C.



ELECTRICAL ENGINEERING RESEARCH LABORATORY
ENGINEERING EXPERIMENT STATION
UNIVERSITY OF ILLINOIS
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1. INTRODUCTION

The study of the ionosphere using the Faraday rotation effect has been undertaken recently by means of rocket, satellite, and moon echo experiments. Different approximations have been used by different authors, resulting in methods with different degrees of complexity, and it is possible to say that the more accurate a method is, the more difficult its application becomes.

However, the use of modern high-speed digital computers offers the possibility of using more complex methods in the solution of this problem.

The program described in this report was written for the ILLIAC, the digital computer of the University of Illinois. Only the general features common to most digital computers will be mentioned.

This program was prepared having in mind the analysis of the Faraday rotation effect, as recorded from artificial satellites. It is intended to be as general as possible in the conditions imposed on the assumed propagating medium: specifically there are no restrictions on the models of the electron density distribution and the earth's magnetic field as long as the ray theory is valid.

The program will trace separately the ordinary and the extraordinary mode, and it will find the virtual phase path length of a ray of each mode between the transmitter (satellite) and a receiver (station).

The difference between respective phase path-lengths is related to the Faraday rotation through a constant which depends on the frequency.

2. GENERAL THEORY OF FARADAY ROTATION

As early as 1845, Faraday had discovered that the plane of polarization of light is rotated through a certain angle when it traverses certain substances in the presence of a longitudinal magnetic field. The same effect has been observed on radio waves traversing the ionosphere or any ionized medium with a magnetic field present, a so-called magneto-ionic medium, even in the case when the magnetic field is not exactly longitudinal.

The reason for this effect is the existence of two characteristic modes of propagation. Each of these modes has its own polarization, phase velocity and group velocity. The polarization and the velocities are completely defined by the characteristics of the medium, so that an originally linearly-polarized wave will be resolved into the two characteristic modes. The total polarization will, therefore, be different at different points in the medium.

If a uniform magneto-ionic medium with a longitudinal field is considered, the characteristic modes are circularly polarized with opposite senses of rotation and slightly different phase velocities. Therefore, the resultant polarization is linear with its plane of polarization determined by the phase relation between the modes. The relation between these phases changes continuously along the ray, resulting in a rotation of the plane of polarization. This rotation is given by (Pedersen, 1927¹).

$$\Omega = \frac{\omega}{2} \left(\frac{1}{v_p(o)} - \frac{1}{v_p(x)} \right) l \quad (2.1)$$

where Ω = rotation of the plane of polarization in radians along a distance l
 l = distance in the phase propagation direction
 $v_p(o)$ = phase velocity of the ordinary mode
 $v_p(x)$ = phase velocity of the extraordinary mode
 ω = angular frequency of the wave

In the ionospheric study using the Faraday effect, none of the conditions mentioned above are present. The radio wave is neither propagating under a strict longitudinal condition nor is the medium homogeneous. Two generalizations are necessary to the original concept of Faraday rotation expressed by (2.1).

When the longitudinal condition is not met, the resultant polarization is generally elliptical. Its shape, as well as its orientation, changes continuously along the propagation direction. It is not exactly a rotation and a careful interpretation should be made. However, (2.1) gives the correct relation between the phases of both modes. From this relation the resultant polarization can be deduced. In the present work it is desirable to generalize the concept of rotation rather than to use the quasi-longitudinal and quasi-transverse approximations. Following this point of view, (2.1) is valid for the general case, and even for the transverse case, if it is decided in the former that the transition from circular to linear polarization is equivalent to a rotation of $\frac{\pi}{2}$ in Ω (Daniels and Bauer, 1959²).

When the medium is not uniform, (2.1) will be written in differential form. For reasons of convenience $\frac{dl}{v_p(o)}$ and $\frac{dl}{v_p(x)}$ will be substituted by $dT^{(o)}$ and $dT^{(x)}$ respectively.

$$d\Omega = \frac{\omega}{2} (dT^{(o)} - dT^{(x)}) \quad (2.2)$$

By integrating this expression between two points T and R:

$$\Omega_{TR} = \frac{\omega}{2} \left[\int_T^R dT^{(o)} - \int_T^R dT^{(x)} \right] \quad (2.3)$$

Where Ω_{TR} is the generalized Faraday rotation between the points T and R and the integrals are the times required for phase propagation of the respective modes between the same two points. This new generalization is made by defining the relation between the phases of both modes. The need of this definition arises from the fact that the characteristic modes follow a different path in a varying medium. However, this expression for Faraday rotation is not arbitrary since it reflects the result that would be obtained if an observation point moved from T to R, investigating the resultant polarization at each point.

In the expression (2.3), it has been assumed that both modes are defined at any point in the integral path as if the medium were uniform in the surroundings of such a point. The assumption is possible only if the change of the physical conditions of the medium is negligible in a distance of a

few wavelengths. Such a medium is called a "slowly varying medium". It is defined in relation to a given frequency. Fortunately, the ionosphere is a "slowly varying" medium with respect to the frequencies used in the Faraday rotation experiments. Furthermore, the integrals can be evaluated independently of one another when both modes propagate independently.

Booker, (1936³) has shown that in a "slowly varying medium," such as the ionosphere, the ordinary and extraordinary mode propagate independently of one another and the ray theory applies to each mode. The polarization ellipse of each mode changes slowly along its path, in accordance with local conditions (electron density, earth's magnetic field and direction of propagation).

Two conditions are listed under which the independence of the modes breaks down and under which coupling between both modes should be considered:

a) In which the refractive indices $\mu^{(o)}$ and $\mu^{(x)}$ approach zero, that is, when the critical plasma frequencies ω^o , ω^x are nearly equal to the frequency considered. In this case the physical conditions vary slowly but the wavelength becomes very large so that the "slowly varying" condition is violated. A wave solution should be sought. Since satellites use frequencies well above the maximum plasma frequency in the ionosphere the refractive index is well above zero. This condition does not occur.

b) Near the "edges" of the ionosphere, where the electron density is low. In this situation the independence of the modes is not preserved. The coupled waves of the other mode, produced over a long part of the ray, have a cumulative effect on the considered ray. The only result is that the characteristic polarization ellipse stops changing at a certain "limiting polarization height". This height is above the lower edge of the ionosphere. This height is dependent on the coupling coefficient, which in turn depends on electron density, magnetic field and collision frequency of electrons (Budden, 1952⁴).

Neither the path of the ray nor the group and phase velocities of either mode are affected because of the coupling in this region. Therefore, the total Faraday rotation defined in (2.3) remains unchanged by this effect.

However, if the polarization of the total wave at the station is examined, this effect should be taken into account since the total wave is formed by the recombination of the ordinary and extraordinary modes as they reach the station,

and the polarization of each mode remains unchanged below the mentioned "limiting polarization height".

The integrals which appear in Equation (2.3) between two points arise from integration along the respective group trajectories or ray paths. Figure 1 illustrates the relation among the differential of length ds along the ray path, the differential time dT , and the phase velocity corresponding to any of the modes.

The phase velocities are given by

$$v_p^{(o)} = \frac{c}{\mu^{(o)}} \quad v^{(x)} = \frac{c}{\mu^{(x)}} \quad (2.4)$$

where c = velocity of light in free space
 $\mu^{(o)}, \mu^{(x)}$ = refractive indices given by the Appleton-Hartree formula (Equation 3.1) and calculated in the wave-normal direction
 By observing Figure 1 and using relations (2.4) it is possible to write:

$$\begin{aligned} dT^{(o)} &= \frac{1}{c} \mu^{(o)} \cos \alpha^{(o)} ds^{(o)} \\ dT^{(x)} &= \frac{1}{c} \mu^{(x)} \cos \alpha^{(x)} ds^{(x)} \end{aligned} \quad (2.5)$$

Equations (2.5) show the convenience of using a vector $\bar{\mu}$ of magnitude μ and with its direction in the direction of the wave normal. The final form of Equation (2.3) will be:

$$\Omega_{TR} = \frac{\omega}{2c} \left[\int_T^R \bar{\mu}^{(o)} \cdot d\bar{s}^{(o)} - \int_T^R \bar{\mu}^{(x)} \cdot d\bar{s}^{(x)} \right] \quad (2.6)$$

Some simplified forms of Equation (2.2) are in common use in the literature. These are given below:

(a) Both modes are assumed to follow the same path as determined by the isotropic refractive index. In this case Equation (2.6) reduces to (Little, and Lawrence, 1960⁵).

$$\Omega = \frac{\omega}{2c} \int (\mu^{(o)} - \mu^{(x)}) ds \quad (2.7)$$

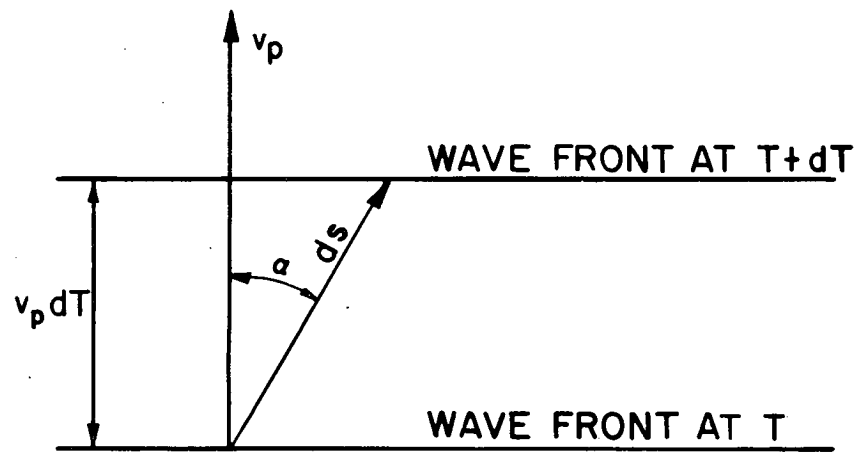


Figure 1. Relation between d_s , dT and v_p

(b) Both modes are assumed to follow a straight line connecting the transmitter and receiver. In this case we can further reduce Equation (2.7) and write it in the following form (Browne, Evans, and Hargreaves, 1959⁶)

$$\Omega = \frac{K}{f^2} \int NMdh \quad (2.8)$$

where

N = electron density

$M = H_L \sec i$ (Yeh and Gonzales, 1960⁷)

dh = differential height

3. THE REFRACTIVE INDEX

The Appleton-Hartree (1932)⁸ formula for the refractive index of a magneto-ionic medium is well known in its physical meaning; however, its use is sometimes obscure. Before proceeding to a presentation of the mathematical method, this topic will be briefly examined. If the collision frequency is disregarded:

$$\left. \begin{array}{l} \mu^{(o)} \\ \mu^{(x)} \end{array} \right\} = 1 - \frac{X}{1 - \frac{Y^2 \sin^2 \Theta}{2(1-X)} \pm \sqrt{\frac{Y^4 \sin^4 \Theta}{4(1-X)^2} + Y^2 \cos^2 \Theta}} \quad (3.1)$$

$\mu^{(o)}$ = refractive index for the ordinary mode, corresponding to the upper sign.

$\mu^{(x)}$ = refractive index for the extraordinary mode, corresponding to the lower sign.

Θ = angle between wave normal and magnetic field.

$$X = \frac{e^2}{4\pi^2 m \epsilon_0} \frac{N}{f^2} \quad (3.2)$$

$$Y = \frac{e \mu_0}{2\pi m} \frac{H}{f}$$

e = charge of an electron (in coulombs)

m = mass of an electron (in kilograms)

ϵ_0 = dielectric constant of free space (MKS)

μ_0 = magnetic permeability of free space (MKS)

f = frequency of wave under consideration (cycles per second)

N = electron density (electrons per cubic meter)

H = intensity of magnetic field (amperes per meter)

The upper index $^{(o)}$ or $^{(x)}$ for the ordinary or extraordinary mode will be omitted on μ and on the other quantities, when the equations could refer to either of the modes.

From the foregoing, it is possible to see that for a given frequency, f , the only quantities not constant are N , H , and Θ . In the steady state, N will be a scalar function of point $(x, y, z, \text{ or } r, \theta, \varphi)$. H will be a vector function of the coordinates, that is, the magnitude and the direction of H will be defined in terms of $x, y, z, \text{ or } r, \theta, \varphi$, according to the coordinate system chosen. Therefore, it is possible to write (3.1) in functional form in the following way:

$$\mu = \mu(x, y, z, \Theta) \quad \text{for rectangular coordinates}$$

or (3.3)

$$\mu = \mu(r, \theta, \varphi, \Theta) \quad \text{for spherical coordinates}$$

The angle Θ is the angle between two directions, that of the magnetic field and that of the wave normal. The direction of the magnetic field is determined by the variables x, y, z (or r, θ, φ). To determine the direction of the wave normal two angles are needed, such as azimuth and altitude, referred to a local orthogonal system centered at the point x, y, z (or r, θ, φ).

It is preferable, instead of two angles, to use the three that the wave normal will form with each of the local orthogonal axes: i_x, i_y, i_z (or i_r, i_θ, i_φ) with an additional condition:

$$\begin{aligned} \cos^2 i_x + \cos^2 i_y + \cos^2 i_z &= 1 \\ \cos^2 i_r + \cos^2 i_\theta + \cos^2 i_\varphi &= 1 \end{aligned} \quad (3.4)$$

In the method that follows, a vector of components v^x, v^y, v^z (or v^r, v^θ, v^φ) referred to a local orthogonal system centered at the point x, y, z , (or r, θ, φ) has been chosen to determine the direction of the wave normal, subject to the following condition:

$$\begin{aligned} v^x^2 + v^y^2 + v^z^2 &= \mu^2 \\ v^r^2 + v^\theta^2 + v^\varphi^2 &= \mu^2 \end{aligned} \quad (3.5)$$

In this way (3.3) can be written in functional form as:

$$\begin{aligned}\mu &= \mu(x, y, z, v^x, v^y, v^z) \\ \mu &= \mu(r, \theta, \varphi, v^r, v^\theta, v^\varphi)\end{aligned}\quad (3.6)$$

From condition (3.5), the following relations are verified:

$$\begin{aligned}v^x &= \mu \cos i_x & v^r &= \mu \cos i_r \\ v^y &= \mu \cos i_y & v^\theta &= \mu \cos i_\theta \\ v^z &= \mu \cos i_z & v^\varphi &= \mu \cos i_\varphi\end{aligned}\quad \text{or} \quad (3.7)$$

For the purposes of the present report and from the last relations, the v 's can be considered as the component of a vector $\bar{\mu}$ of magnitude μ . However, it should be remembered that their principal function is to define a direction.

A surface on which the refractive index in a given direction is constant is defined as a "stratification surface". For example:

$$\begin{aligned}\mu(x, y, z, v^x, v^y, v^z) &= \text{const} \\ v_x &= \text{const} \\ v_y &= \text{const} \\ v_z &= \text{const}\end{aligned}\quad (3.8)$$

In Section 5 it will be shown that Snell's law is verified with respect to these stratification surfaces in the differential ray tracing Equations (4.1). The surfaces (3.8) are not the same as:

$$\begin{aligned}\mu(r, \theta, \varphi, v^r, v^\theta, v^\varphi) &= \text{const} \\ v^r &= \text{const} \\ v^\theta &= \text{const} \\ v^\varphi &= \text{const}\end{aligned}\quad (3.9)$$

Since the same set v^r, v^θ, v^φ define a different direction in each point of space, the partial derivatives $\left(\frac{\partial \mu}{\partial x} \right)_{y, z, v^x, v^y, v^z}, \left(\frac{\partial \mu}{\partial y} \right)_{x, z, v^x, v^y, v^z}, \left(\frac{\partial \mu}{\partial z} \right)_{x, y, v^x, v^y, v^z}$

will represent the components of a vector perpendicular to the stratification surfaces with similar characteristics to those of a gradient. These partial derivatives would represent the variation of μ in space when μ is considered in the same direction at every point. The corresponding relations in spherical coordinates do not have the same characteristics because a change of the coordinates θ and φ with the v 's constant implies a change of direction in space for μ . For example, if a uniform medium is considered, i.e., a medium with constant electron density and constant magnetic field (in magnitude and direction) through all space, then:

$$\left(\frac{\partial \mu}{\partial x} \right)_{y,z,v^x,v^y,v^z} = \left(\frac{\partial \mu}{\partial y} \right)_{x,z,v^x,v^y,v^z} = \left(\frac{\partial \mu}{\partial z} \right)_{x,y,v^x,v^y,v^z} = 0 \quad (3.10)$$

But if a spherical coordinate system is chosen:

$$\left(\frac{\partial \mu}{\partial r} \right)_{\theta,\varphi,v^r,v^\theta,v^\varphi} = 0; \quad \left(\frac{\partial \mu}{\partial \theta} \right)_{r,\varphi,v^r,v^\theta,v^\varphi} \neq 0; \quad \left(\frac{\partial \mu}{\partial \varphi} \right)_{r,\theta,v^r,v^\theta,v^\varphi} \neq 0 \quad (3.11)$$

A final point to consider is the fact that the stratification surfaces do not correspond to surfaces of constant electron density, unless the magnetic field is constant over the surface.

4. MATHEMATICAL METHOD

4.1 Ray Differential Equations

The ray tracing equations used are those derived from Hamilton's method and expressed in general, rectangular and spherical coordinates by Haselfrove (1955)⁹.

The following set of equations for spherical coordinates was chosen:

$$\begin{aligned} \frac{dr}{dt} &= \frac{1}{\mu^2} (v^r - \mu \frac{\partial \mu}{\partial v^r}) \\ \frac{d\theta}{dt} &= \frac{1}{r\mu^2} (v^\theta - \mu \frac{\partial \mu}{\partial v^\theta}) \\ \frac{d\varphi}{dt} &= \frac{1}{r \sin \theta \mu^2} (v^\varphi - \mu \frac{\partial \mu}{\partial v^\varphi}) \end{aligned} \tag{4.1}$$

$$\frac{dv^r}{dt} = \frac{1}{\mu} \frac{\partial \mu}{\partial r} + v^\theta \frac{d\theta}{dt} + \sin \theta v^\varphi \frac{d\varphi}{dt}$$

$$\frac{dv^\theta}{dt} = \frac{1}{r} \left\{ \frac{1}{\mu} \frac{\partial \mu}{\partial \theta} - v^\theta \frac{dr}{dt} + r \cos \theta v^\varphi \frac{d\varphi}{dt} \right\}$$

$$\frac{dv^\varphi}{dt} = \frac{1}{r \sin \theta} \left\{ \frac{1}{\mu} \frac{\partial \mu}{\partial \varphi} - \sin \theta v^\varphi \frac{dr}{dt} - r \cos \theta v^\varphi \frac{d\theta}{dt} \right\}$$

where:

μ = refractive index in the wave normal direction

r, θ, φ = spherical coordinates referred to a system with its origin at the center of the Earth. The z axis passes through the north pole. The Greenwich meridian is in the xz plane.

v^r, v^θ, v^φ = components of $\bar{\mu}$ along the local orthogonal system associated with the point (r, θ, φ)

t = virtual phase path length. This variable is related to the time of phase propagation along the ray path by the velocity of light c .

The integration of these equations gives the trajectory of the "group" path or "ray" corresponding to the "o" or "x" mode according to the chosen value of the refractive index. The final value of t is the phase length along the ray.

The partial derivatives $\frac{\partial \mu}{\partial v^r}$, $\frac{\partial \mu}{\partial v^\theta}$, $\frac{\partial \mu}{\partial v^\varphi}$ have to be transformed into a more workable form since their meaning is somewhat obscure.

The functional relation of μ with respect to v^r , v^θ , and v^φ can be expressed:

$$\mu = \mu [r, \theta, \varphi, \Theta]$$

$$\begin{aligned} \Theta &= \Theta(v^r, v^\theta, v^\varphi) \\ &= \cos^{-1} \frac{v^r \cos \eta_r + v^\theta \cos \eta_\theta + v^\varphi \cos \eta_\varphi}{\sqrt{v^r{}^2 + v^\theta{}^2 + v^\varphi{}^2}} \end{aligned} \quad (4.2)$$

where $\cos \eta_r$, $\cos \eta_\theta$, $\cos \eta_\varphi$ = direction cosines of the earth's magnetic field referred to the local orthogonal system associated with the point (r, θ, φ) .

From Equations (4.2):

$$\begin{aligned} \frac{\partial \mu}{\partial v^r} &= \frac{\partial \mu}{\partial \Theta} \frac{\partial \Theta}{\partial v^r} \\ \frac{\partial \mu}{\partial v^\theta} &= \frac{\partial \mu}{\partial \Theta} \frac{\partial \Theta}{\partial v^\theta} \\ \frac{\partial \mu}{\partial v^\varphi} &= \frac{\partial \mu}{\partial \Theta} \frac{\partial \Theta}{\partial v^\varphi} \end{aligned} \quad (4.3)$$

Performing operations (4.3) on expressions (4.2) and substituting into the first three of equations (4.1),

$$\frac{dr}{dt} = \frac{1}{\mu} \left[v^r - D (v^r \cos \Theta - \mu \cos \eta_r) \right] \quad (4.4)$$

$$\frac{d\theta}{dt} = \frac{1}{r\mu} [v^\theta - D (v^\theta \cos \Theta - \mu \cos \eta_\theta)] \quad (4.4)$$

$$\frac{d\varphi}{dt} = \frac{1}{r \sin \theta \mu^2} [v^\varphi - D (v^\varphi \cos \Theta - \mu \cos \eta_\varphi)]$$

where D is defined as:

$$D = \frac{1}{\mu \sin \Theta} \frac{\partial \mu}{\partial \Theta} \quad (4.5)$$

$$= \frac{Y^2 \cos \Theta}{\sqrt{\frac{Y^4 \sin^4 \Theta}{4(1-X)^2} + Y^2 \cos \Theta}} \frac{1 - \mu^2}{2(1-X)} \quad \begin{cases} + \text{ for } (o) \\ - \text{ for } (x) \end{cases}$$

The set of differential equations used to obtain the ray are set (4.4) and the last three equations of (4.1).

Condition (3.5), it is noticed, would permit reduction of the last three equations (4.1) to only two. However, the difference between the value of μ obtained from the integrated values of v^r , v^θ , v^φ and that obtained directly by the Appleton-Hartree formula offers a method of checking the accuracy of the integrations at all points.

For the purpose of integration in the computer a seventh equation is added:

$$\frac{dt}{dt} = 1 \quad (4.6)$$

4.2 Initial Wave Normal Direction

If the initial values of r , θ , φ , v^r , v^θ , and v^φ are known, the integration can be performed. In the case of a satellite transmitting and a ground station receiving, the values known are the position of the satellite $(r_o, \theta_o, \varphi_o)$ which will be called point T, and the position of the station $(r_1, \theta_1, \varphi_1)$ which will be called point R.

A direct calculation of v^r , v^θ , v^φ at point T is difficult, but if the direction cosines α , β , γ of a vector in the initial ray direction referred

to the local orthogonal axis are known, v^r , v^θ , and v^φ can be calculated from them in the following way:

$$v_{pg} = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r \frac{d\theta}{dt}\right)^2 + \left(r \sin \theta \frac{d\varphi}{dt}\right)^2} \quad (4.7)$$

where v_{pg} is related to the phase velocity in the direction of the ray by the velocity of light.

$$\frac{1}{v_{pg}} \frac{dr}{dt} = \alpha$$

$$\frac{1}{v_{pg}} r \frac{d\theta}{dt} = \beta \quad (4.8)$$

$$\frac{1}{v_{pg}} r \sin \theta \frac{d\varphi}{dt} = \gamma$$

Substituting (4.4) into (4.7)

$$v_{pg} = \frac{\sqrt{1 + D^2 \sin^2 \Theta}}{\mu} \quad (4.9)$$

Substituting (4.9) and (4.4) into (4.8) and solving for v^r , v^θ , and v^φ :

$$\begin{aligned} v^r &= \mu \left[\frac{\alpha(1 + D^2 \sin^2 \Theta)^{1/2} - D \cos \eta_r}{1 - D \cos \Theta} \right] \\ v^\theta &= \mu \left[\frac{\beta(1 + D^2 \sin^2 \Theta)^{1/2} - D \cos \eta_\theta}{1 - D \cos \Theta} \right] \\ v^\varphi &= \mu \left[\frac{\gamma(1 + D^2 \sin^2 \Theta)^{1/2} - D \cos \eta_\varphi}{1 - D \cos \Theta} \right] \end{aligned} \quad (4.10)$$

There is still one unknown in set (4.10) i.e., Θ . This can be found by substituting (4.10) into the second set of Equations (4.2) and simplifying:

$$\cos \Theta = \frac{\Lambda(1 + D^2 \sin^2 \Theta)^{1/2} - D}{1 - D \cos \Theta} \quad (4.11)$$

where Λ is defined by

$$\Lambda = \alpha \cos \eta_r + \beta \cos \eta_\theta + \gamma \cos \eta_\varphi \quad (4.12)$$

$\cos \Theta$ can be determined from (4.11) by an iterative method taking advantage of the fact that for frequencies fairly large $\cos \Theta \cong \Lambda$

After $\cos \Theta$ and $|\sin \Theta|$ are known, v^r , v^θ and v^φ can be calculated from α, β and γ using Equations (4.10).

4.3 Initial Ray Direction

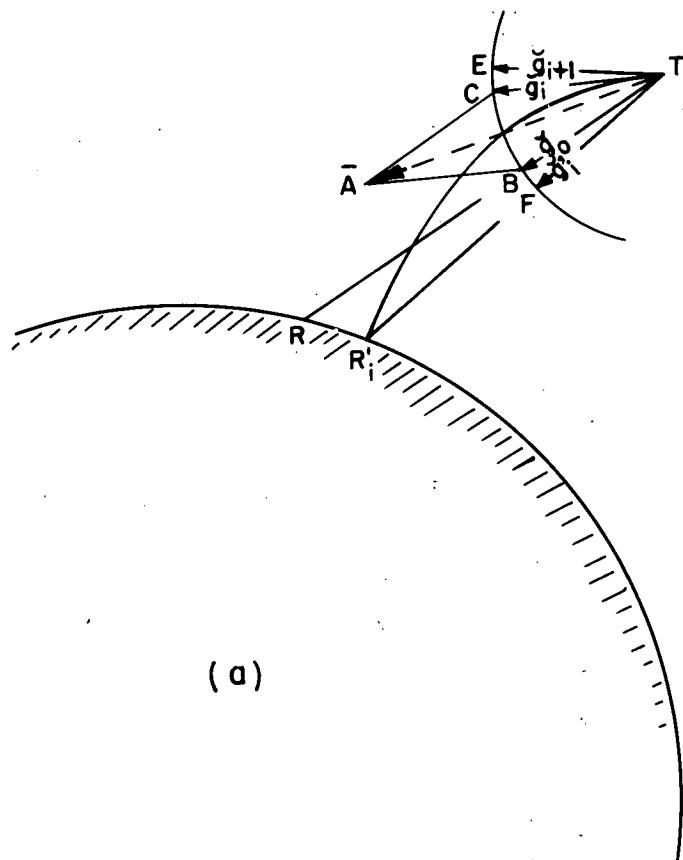
The problem now is how to calculate α , β and γ at point T in such a way that the ray will pass through point R. Since this solution is intended to be independent on the electron density distribution and the earth's magnetic field models used, no mathematical derivation will be of help since it would be different for each mode 1. In addition it would be extremely complex. A successive approximation method can be used.

A ray integrated with the values α_i , β_i , γ_i at point T generally will reach the sphere $r = r_1$, with coordinated θ and φ differing from those of point R, but at a point R'_i . A correction applied to the values α_i , β_i , γ_i , considering the result obtained, will yield the values α_{i+1} , β_{i+1} , γ_{i+1} . These values will be more successful in approaching point R.

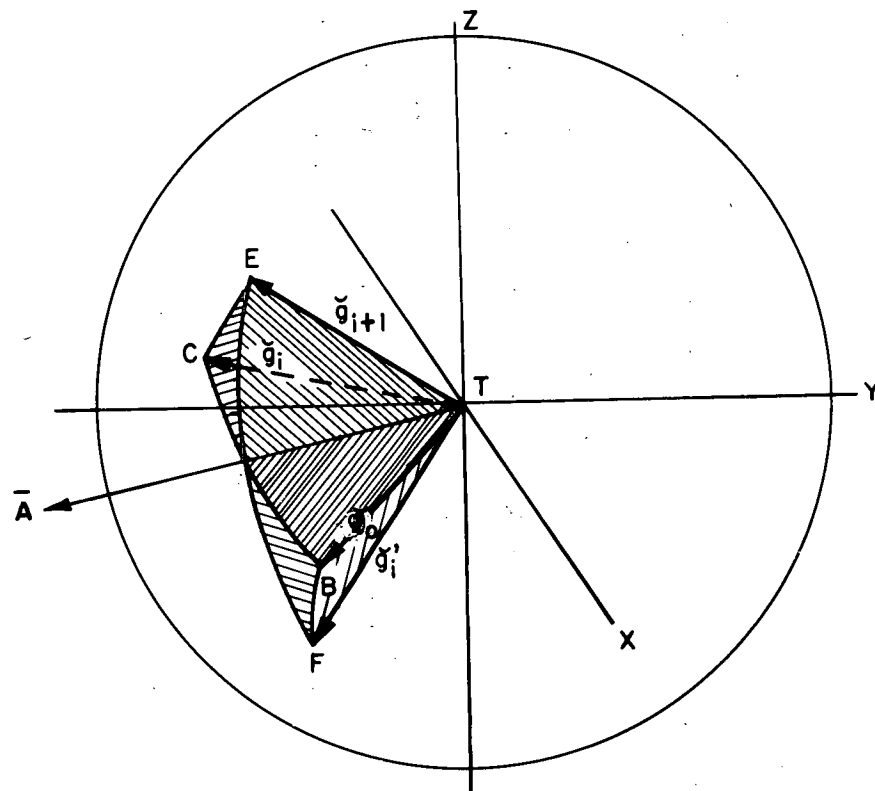
This is explained in Figure 2a where:

- \vec{g}_0 = unit vector in TR direction of components α_0 , β_0 , γ_0
- \vec{g}_i , \vec{g}_{i+1} = unit vectors of components α_i , β_i , γ_i and α_{i+1} , β_{i+1} , γ_{i+1}
which would correspond to trials of order i and i+1
- \vec{g}'_i = unit vector in the TR'_i direction with components α'_i , β'_i , γ'_i .
- R'_i = the point at which the ray obtained with the values α_i , β_i , γ_i
reaches the condition $r = r_1$

In this instance, a good correction when a plane case is considered, would



(a)



(b)

Figure 2. Relationship between two successive values of g_i

be to take \vec{g}_{i+1} as shown in Figure 2, specifically, the angle between \vec{g}'_i and \vec{g}_0 equal to the angle between \vec{g}_i and \vec{g}_{i+1} , or if a vector \vec{A} is defined:

$$\vec{A} = \vec{g}_i + \vec{g}_0 \quad (4.13)$$

The correction explained above would make the points F and B symmetrical with points E and C in respect to the direction of the vector \vec{A} .

This latter criterion can be used in the three dimensional case to correct the initial ray direction used in trial of order i , as it is explained in Figure 2b. The symmetry of F and E with respect to the direction of \vec{A} can be expressed:

$$\vec{g}_{i+1} + \vec{g}'_i = K\vec{A} \quad (4.14)$$

$$\vec{g}_{i+1} \cdot \vec{A} = \vec{g}'_i \cdot \vec{A}$$

where K is a scalar constant of proportionality which can be calculated by substituting \vec{g}_{i+1} from the first of (4.14) into the second:

$$K = \frac{2 \vec{g}'_i \cdot \vec{A}}{\vec{A} \cdot \vec{A}} \quad (4.15)$$

Substituting (4.13) into the first Equation (4.14) and expressing the equation obtained, as well as (4.15), in terms of α 's, β 's and γ 's,

$$K = \frac{\alpha'_i (\alpha_i + \alpha_0) + \beta'_i (\beta_i + \beta_0) + \gamma'_i (\gamma_i + \gamma_0)}{1 + \alpha_i \alpha_0 + \beta_i \beta_0 + \gamma_i \gamma_0}$$

$$\alpha_{i+1} = K (\alpha_i + \alpha_0) - \alpha'_i$$

(4.16)

$$\beta_{i+1} = K (\beta_i + \beta_0) - \beta'_i$$

$$\gamma_{i+1} = K (\gamma_i + \gamma_0) - \gamma'_i$$

The first trial can be the values α_o , β_o and γ_o .

The following formulae, which give the components of the unit vector on the direction between two points T and R, at point T, can be easily derived from geometric consideration:

$$d = \sqrt{1 + \left(\frac{r_1}{r_o}\right)^2 - 2 \frac{r_1}{r_o} [\sin \theta_1 \sin \theta_o \cos (\varphi_o - \varphi_1) + \cos \theta_1 \cos \theta_o]}$$

$$\alpha_o = \frac{1}{d} \left\{ 1 - \frac{r_1}{r_o} [\sin \theta_1 \sin \theta_o \cos (\varphi_o - \varphi_1) + \cos \theta_1 \cos \theta_o] \right\} \quad (4.17)$$

$$\beta_o = \frac{1}{d} \left\{ \frac{r_1}{r_o} [\sin \theta_o \cos \theta_o \cos (\varphi_1 - \varphi_o) - \cos \theta_1 \sin \theta_o] \right\}$$

$$\gamma_o = \frac{1}{d} \left[\frac{r_1}{r_o} \sin \theta_1 \sin (\varphi_1 - \varphi_o) \right]$$

Equations (4.17) are also used when calculating α'_i , β'_i , γ'_i using the appropriate values for θ' and φ' .

4.4. Coordinates

The ability to give the coordinates of points along a ray, in order to plot a ray if desired, is a desirable feature of this program. This feature involves the capability of obtaining points along the ray referred to a convenient coordinate system. A local spherical system has been chosen, centered at the center of the Earth with the z' axis passing through the receiving station S and the $x' z'$ plane passing through the satellite T.

In this way r remains untransformed but the difference between r and the radius of the Earth a results in the height above the ground. The coordinate θ' gives the angular distance from the station. The value φ' gives the deviation of the ray from the vertical plane which contains T and R. Furthermore, since the lateral deviation is small, multiplying φ' by $a \sin \theta'$ the deviation can be expressed in linear units perpendicular to the $x' z'$ plane and measured on the surface of the Earth.

It is convenient to use an intermediate coordinate system to transform the coordinates. This system is taken with the z'' axis coincided with z' and the $x'' z''$ plane containing the x axis. In this case

$$\begin{aligned} z' &= z'' \\ \theta' &= \theta'' \\ \varphi' &= \varphi'' - \varphi''_0 \end{aligned} \quad (4.18)$$

where φ''_0 is the coordinate φ'' of the satellite. The transformation from the original to the intermediate coordinate system can be easily obtained.

$$\begin{aligned} \cos \theta'' &= \cos \theta \cos \theta_1 + \sin \theta \sin \theta_1 \cos (\varphi - \varphi_1) \\ \sin \varphi'' &= \frac{\sin \theta \sin (\varphi - \varphi_1)}{\sin \theta''} \\ \cos \varphi'' &= \frac{\cos \theta - \cos \theta_1 \cos \theta''}{\sin \theta_1 \sin \theta''} \end{aligned} \quad (4.19)$$

5. DISCUSSION OF THE RAY TRACING EQUATIONS

The differential equations used in the integration are the set (4.4), (4.6) and the last three Equations of (4.1). Interpreting the variables in these equations is somewhat confusing, particularly the variable t , therefore, a brief discussion, with a special case, may add understanding to its meaning.

The point, A, and B shown in Figure 3 will be examined. Point A is associated with the set of values (r, θ, φ, t) and point B is associated with the set of values $(r + \Delta r, \theta + \Delta \theta, \varphi + \Delta \varphi, t + \Delta t)$. For discussion purposes let us choose point A on the ray such that μ is in the r -direction and the magnetic field is in the $r \theta$ plane. These conditions are expressed as:

$$\begin{aligned} v^r &= \mu \\ v^\theta &= v^\varphi = 0 \\ \Theta &= \eta_r \\ \cos \eta_\varphi &= 0 \end{aligned} \tag{5.1}$$

Replacing the variable t by T :

$$t = c T \tag{5.2}$$

and taking finite differences in Equations (4.4) according to the conditions expressed in (5.1), the following results are obtained:

$$\begin{aligned} \Delta r &= \frac{c}{\mu} \Delta T \\ r \Delta \theta &= \frac{c}{\mu} \frac{\partial \mu}{\partial \Theta} \Delta T \\ \Delta \varphi &= 0 \end{aligned} \tag{5.3}$$

The fact that point B belongs to the group trajectory can be checked by investigating the angle α between the wave normal and the obtained Δs .

$$\tan \alpha = \frac{r \Delta \theta}{\Delta r} = \frac{1}{\mu} \frac{\partial \mu}{\partial \Theta} \tag{5.4}$$

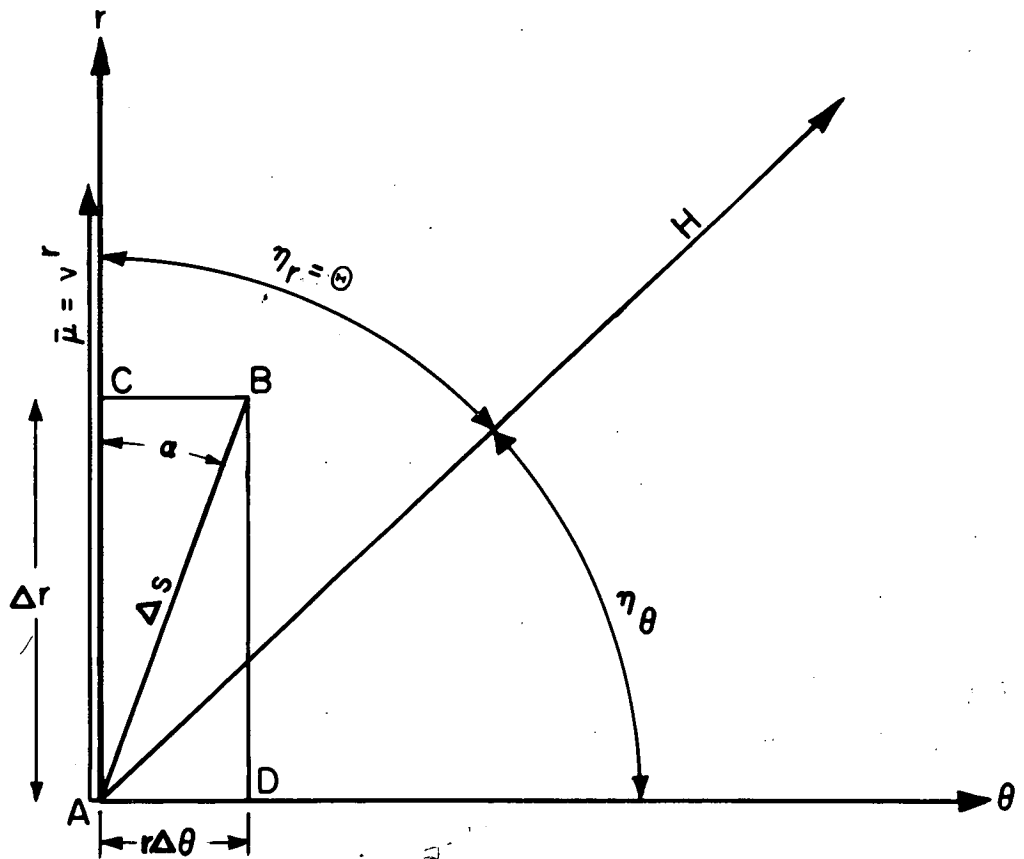


Figure 3. Relationship between Δs and Δr when the phase velocity coincides with the r direction.

The last expression is a well-known relation between wave normal and group velocity directions (Bremmer, 1949¹⁰). The first Equation (5.3) relates Δr and ΔT by the familiar phase velocity. The wave front AD will move to CB in the time ΔT , or the wave front moves a length Δs along the group trajectory in the time ΔT . This last interpretation and Equation (5.2) justifies the definition for the variable t .

From Figure 3 it can also be seen that the relations (2.5) are satisfied.

The Faraday rotation can be expressed in terms of the variable t in the following way:

$$\begin{aligned}\Omega &= \frac{\omega}{2c} (t_o - t_x) && \text{radians} \\ \Omega &= \frac{f}{2c} (t_o - t_x) && \text{turns}\end{aligned}\tag{5.5}$$

The Snell law is the second point to be discussed. It has been shown that in a rectangular orthogonal system of reference, the derivatives $\frac{\partial \mu}{\partial x}$, $\frac{\partial \mu}{\partial y}$, $\frac{\partial \mu}{\partial z}$ can be considered the components of a vector perpendicular to the stratification surfaces, and because of this property the rectangular coordinates will be used in this discussion, leaving to Appendix II the demonstration of the equivalence of the set of differential equations in both coordinate systems.

The equivalent differential equations in rectangular coordinates to the last three Equations (4.1) are:

$$\begin{aligned}\frac{dv^x}{dt} &= \frac{1}{\mu} \frac{\partial \mu}{\partial x} \\ \frac{dv^y}{dt} &= \frac{1}{\mu} \frac{\partial \mu}{\partial y} \\ \frac{dv^z}{dt} &= \frac{1}{\mu} \frac{\partial \mu}{\partial z}\end{aligned}\tag{5.6}$$

Taking finite differences between two points it follows from (5.6) that

$$\frac{\Delta v^x}{\frac{\partial \mu}{\partial x}} = \frac{\Delta v^y}{\frac{\partial \mu}{\partial y}} = \frac{\Delta v^z}{\frac{\partial \mu}{\partial z}}\tag{5.7}$$

Equation (5.7) shows that $\Delta\bar{\mu}$ is parallel with the normal \vec{n}_1 , to the stratification surfaces in the vicinity of the points considered. The situation is illustrated in Figure 4. Snell's law can be written:

$$\mu_1 \sin i_1 = \mu_2 \sin i_2 \quad (5.8)$$

It is seen that the stratification surfaces have been assumed constant between points 1 and 2, separated by Δs . The direction of the normal to this surface is determined by the value of the partial derivatives $\frac{\partial \mu}{\partial x}$, $\frac{\partial \mu}{\partial y}$ and $\frac{\partial \mu}{\partial z}$ evaluated at point 1, or between points 1 and 2. If a third point is considered, the partial derivatives will have, in general, a different value and the normal \vec{n}_2 between points 2 and 3 will be different from \vec{n}_1 . This is why differential equations used in this method make no assumption as to the electron density distribution or the magnetic field model. In particular, no assumptions about horizontal or spherical stratification are needed.

A limitation in Equations 4.6 arises when θ approaches 0° , or 180° . $\frac{d\varphi}{dt}$ and $\frac{dv\varphi}{dt}$ tend to ∞ , because of the presence of $\sin \theta$ in the denominator of these derivatives. The physical meaning is easy to understand because in a ray passing through the z axis the quantity φ increases abruptly by π . The derivative becomes infinite at these points.

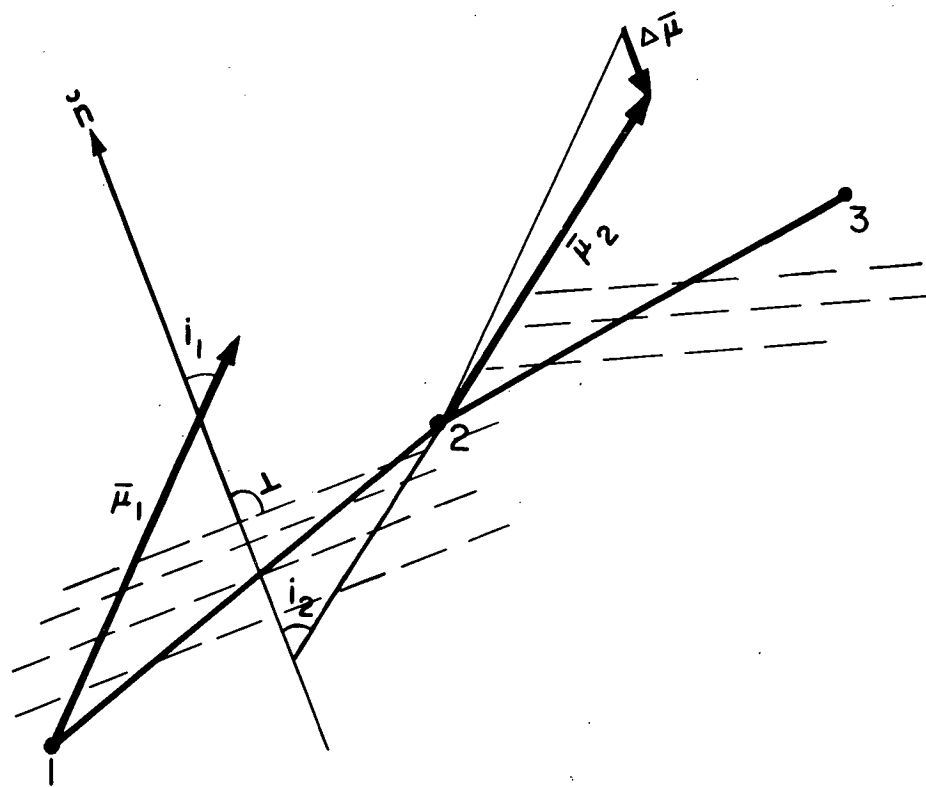


Figure 4. Snell's law between points 1 and 2

6. GENERAL DESCRIPTION OF PROGRAM

The general description of the program is given by the flow diagrams of Figures 5, 6, and 8. These diagrams will be explained in this section following a brief discussion concerning the integration method.

The differential ray equations are integrated using the ILLIAC Library subroutines (F1-Wheeler, 1953¹¹) and (F5-Muller, 1956¹²). Both subroutines use the Runge-Kutta or Gill-Kutta method of integration of a system of ordinary differential equations (Fosdick, 1958¹³ and Gill, 1951¹⁴). They require an "auxiliary subroutine" which will calculate the following quantities:

$$\begin{aligned}
 &2^m \frac{dr}{dt} \Delta t, \quad 2^m \frac{d\theta}{dt} \Delta t, \quad 2^m \frac{d\varphi}{dt} \Delta t \\
 &2^m \frac{dv^r}{dt} \Delta t, \quad 2^m \frac{dv^\theta}{dt} \Delta t, \quad 2^m \frac{dv^\varphi}{dt} \Delta t \\
 &2^m \frac{dt}{dt} \Delta t = 2^m \Delta t
 \end{aligned} \tag{6.1}$$

where

Δt = Length of integration step

2^m = A factor introduced to increase the accuracy of the integration.

This factor is chosen considering the scaling of all the intervening quantities and the range of values that they can take.

The difference between both subroutines is as follows:

When an entry is made to the subroutine (F5) in a part of the program, the subroutine will integrate the equations until one of the variables reaches a given value before the control is returned to the main program. This means (F5) will integrate as many integral steps Δt as required. It will also determine the final fraction of Δt to adjust the given variable to the given value.

Subroutine (F5) is used to integrate the differential equations up to the surface of the Earth, which corresponds to a value of r equal to the radius of the Earth.

When an entry is made to the subroutine (F1), the subroutine will integrate one integral step Δt and will return the control to the main program. This subroutine is used only when the coordinates of points along the ray are required. The last point does not coincide in general with the point R.

6.1 General Flow Diagram

This diagram (Figure 5) shows the operation of input and the interpretation of orders in the data tape.

When using the program, the first step is to read the program in the ILLIAC in the usual manner. The ILLIAC will stop in a "black switch stop" (B. S. S.). As soon as the desired portion of the data tape is set in the reader, the program will read the corresponding section of data by moving the "black switch up". Unless the data corresponding to the ray sought is the one in question, the program will stop again in a B. S. S. ready to read another section of data. The data of magnetic field, electron density, and frequency can be read in any order provided the proper order is heading the data. This will be explained in a future section.

When the "ray sought data" is read, the program follows another pattern: it prints the coordinates of the points T and R and it begins to integrate the ordinary or extraordinary ray, as desired. After obtaining the extraordinary ray, the program is ready to read a new set of data.

6.2 Integration Flow Diagram

Figure 6 shows how the program makes successive approximations of initial conditions. With the first trial values of the initial direction cosines $\alpha_o, \beta_o, \gamma_o$, the program calculates and integrates the corresponding values of v_r, v_θ, v_ϕ . After the integration is completed up to the surface of the Earth, it tests whether the quantities $|\varphi_o - \varphi'_o|$ and $|\theta_o - \theta'_o|$ are small enough. If they are not it calculates $\alpha_1, \beta_1, \gamma_1$ and repeats the cycle as many times as necessary.

Figure 7 shows an example of the values $|\varphi_o - \varphi'_i|$ and $|\theta_o - \theta'_i|$ versus the trial number i . It can be seen that it follows an exponential law until $|\varphi_o - \varphi'_i|$ and $|\theta_o - \theta'_i|$ approach the limit of the computer's accuracy. In the exponential region we can write.

$$\begin{aligned} |\theta_o - \theta'_i| &= |\theta_o - \theta'_o| e^{-B_1 i} \\ |\varphi_o - \varphi'_i| &= |\varphi_o - \varphi'_o| e^{-B_2 i} \end{aligned} \quad (6.2)$$

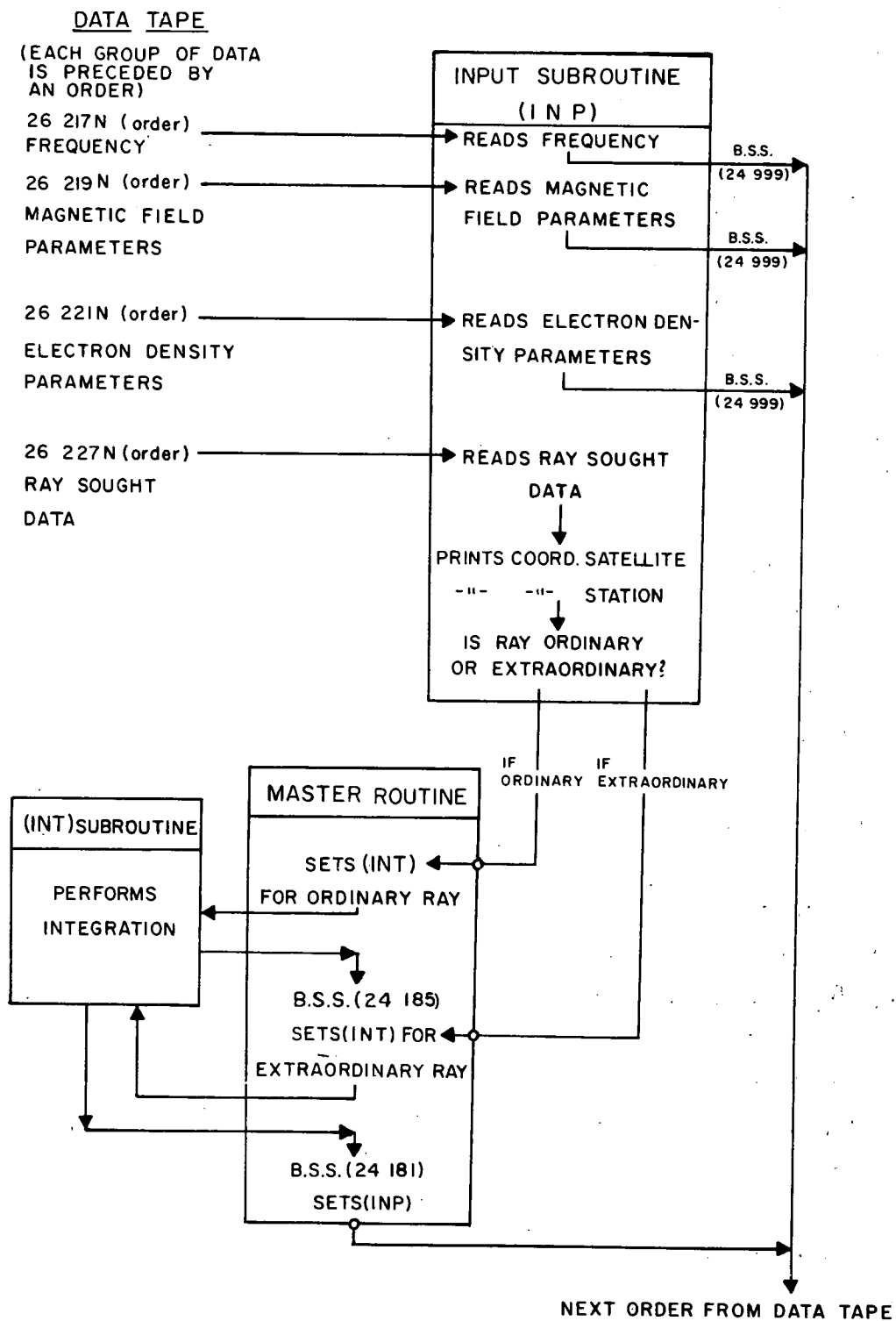


Figure 5. General flow diagram

Note: B.S.S. means "black switch stop": The ILLIAC stops. By moving up the "black switch" it continues working.

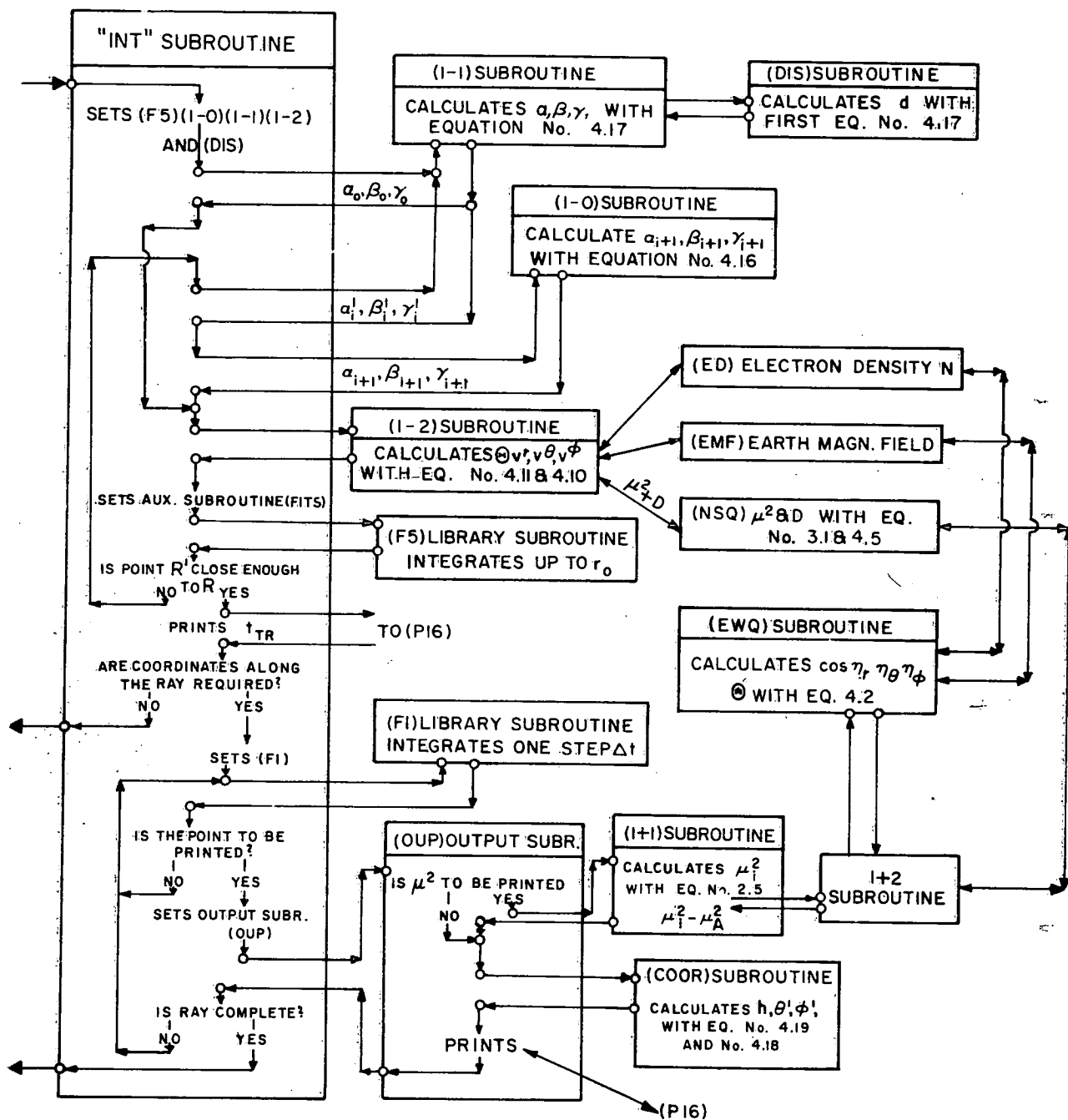


Figure 6. Integration flow diagram

Note: (P16) is a library subroutine for punching out numbers

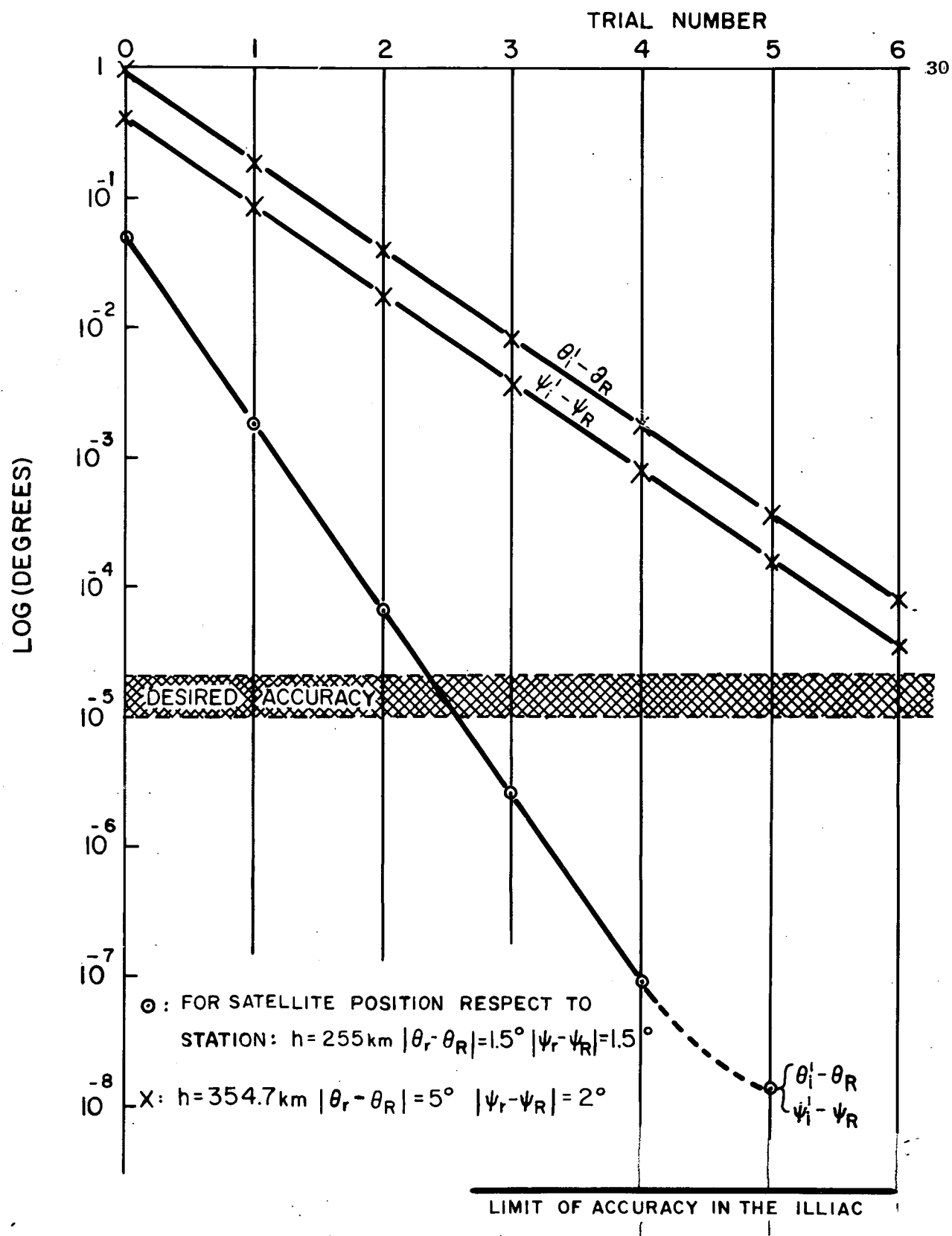


Figure 7. $|\theta'_i - \theta_r|$ and $|\varphi'_i - \varphi_r|$ as a function of trial number.

where the constants B_1 and B_2 are proportional to the slope of the corresponding curves in Figure 8. The B 's depend on many factors, such as the distance between T and R, the electron density distribution, and to a lesser extent, the magnetic field variations between T and R.

When the distance between R and R' is satisfactorily small, the program will print the obtained value of t . If points along the ray are required it will also integrate the ray again by using the initial values v^r, v^θ, v^φ used in the last trial. This integration will be carried out with the subroutine (F1) and it will print out points along the ray and the corresponding values of the error.

$$\frac{1}{2} \epsilon = \frac{1}{2} (\mu_i^2 - \mu_A^2) \quad (6.3)$$

where μ_i^2 = refractive index calculated from expression No. 3.5 using the integrated values of v^r, v^θ , and v^φ .

μ_A^2 = refractive index calculated from Appleton-Hartree formula in the direction defined by the values v^r, v^θ, v^φ used to calculate μ_i

It will be noted that subroutine (1-2) needs the use of the subroutines (ED) (EMF) and (NSQ); so does the subroutine (EWQ). The last subroutine is explained in more detail in Figure 7.

6.3. Auxiliary Subroutine Flow Diagram

The function of this subroutine has been explained as complementary to (F1) and (F5).

The flow diagram corresponding to this subroutine is given in Figure 8. It can be seen that the subroutine (EWQ) requires the use of (ED) and (EMF). (EWQ) calculates, then, $\cos \Theta$ and $\sin \Theta$ as the angle between the phase velocity direction defined by v^r, v^θ and v^φ and the magnetic field. With the values of $\cos \Theta$ and $\sin \Theta$ calculated in this way and those of N and H, (NSQ) calculates μ^2 and D using Equation Nos. (3.1) and (4.5). The final result of an entry to subroutine (1-2) with the values of $r, \theta, \varphi, v^r, v^\theta$ and v^φ is to obtain μ^2 and D or μ^2 only from the local physical conditions. It should be noted that (1-2) uses the set v^r, v^θ and v^φ only as a direction.

The subroutine (DER) calculates $\frac{\partial \mu}{\partial r}, \frac{1}{r} \frac{\partial \mu}{\partial \theta}$, and $\frac{1}{r \sin \theta} \frac{\partial \mu}{\partial \varphi}$ by taking finite differences. If the values $\Delta_1 r, \Delta_1 \theta$ and $\Delta_1 \varphi$ are given:

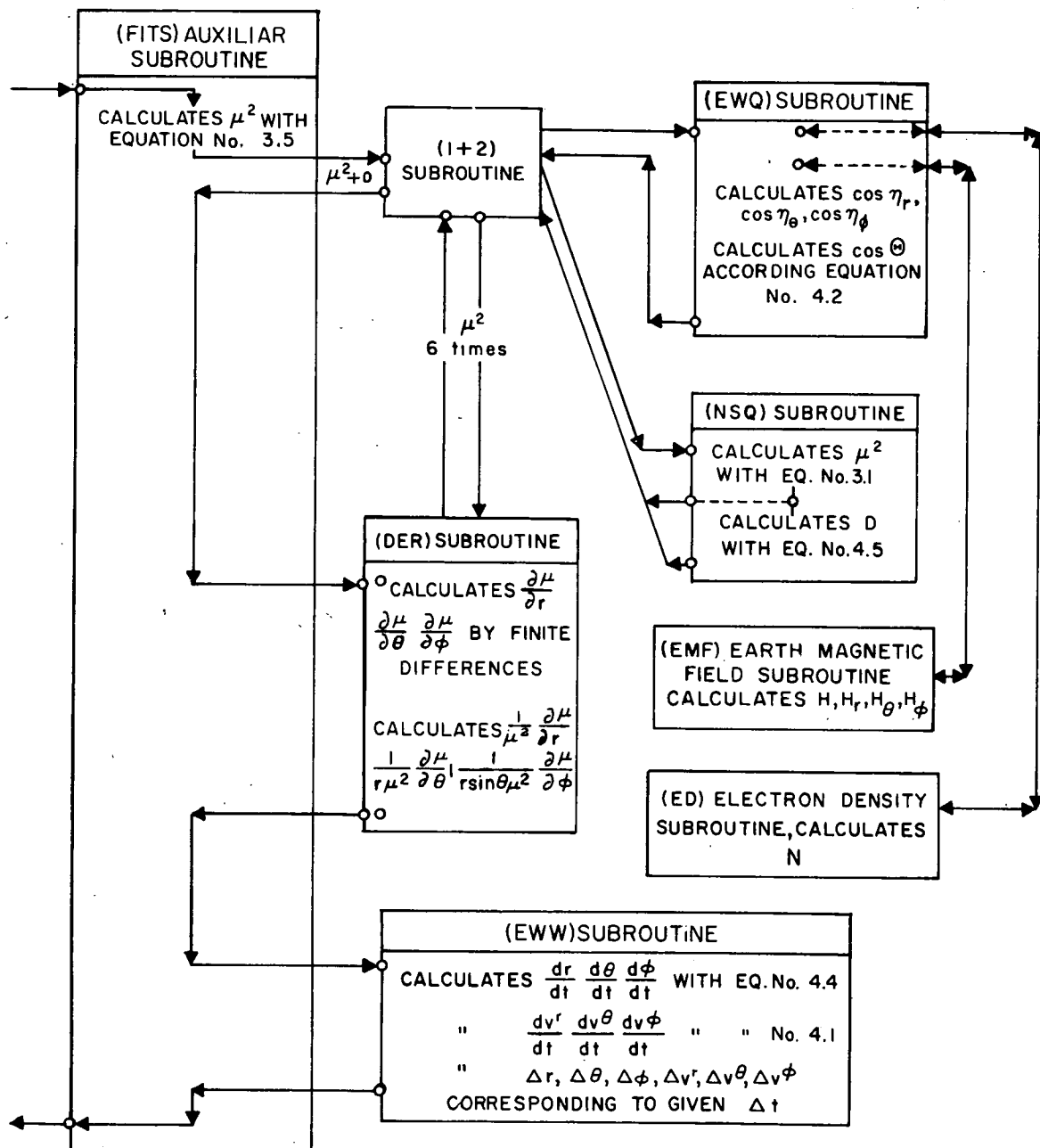


Figure 8. Auxiliary subroutine flow diagram

$$\frac{\partial \mu}{\partial r} = \frac{1}{2\mu} \frac{\partial(\mu^2)}{\partial r} = \frac{\mu^2(r + \frac{\Delta_1 r}{2}, \theta, \varphi) - \mu^2(r - \frac{\Delta_1 r}{2}, \theta, \varphi)}{2\mu \Delta_1 r}$$

$$\frac{1}{r} \frac{\partial \mu}{\partial \theta} = \frac{\mu^2(r_1 \theta + \frac{\Delta_1 \theta}{2}, \varphi) - \mu^2(r_1 \theta - \frac{\Delta_1 \theta}{2}, \varphi)}{2r \mu \Delta_1 \theta} \quad (6.4)$$

$$\frac{1}{r \sin \theta} \frac{\partial \mu}{\partial \varphi} = \frac{\mu^2(r_1 \theta_1 \varphi + \frac{\Delta_1 \varphi}{2}) - \mu^2(r_1 \theta_1 \varphi - \frac{\Delta_1 \varphi}{2})}{2\mu r \sin \theta \Delta_1 \varphi}$$

where v^r, v^θ, v^φ remain constant.

This method of taking a derivative is equivalent to considering a second degree curve passing through every set of three points: $(r + (\Delta_1 r)/2, \theta, \varphi)$, (r, θ, φ) and $(r - (\Delta_1 r)/2, \theta, \varphi)$ and similarly in the θ and φ directions.

The advantage of the method is that it can be used with any model of electron density and magnetic field. However, it has been found that to keep the error small, the second partial derivatives of N have to be continuous: This is depicted in Figure 9, where the error ϵ (Equation No. (6.3) is shown versus height for a vertical ray in a spherically stratified isotropic medium with assumed "smooth" parabolic electron distribution.. The smooth parabola has been obtained by rounding off parabolic distribution with additional parabolas in the edges. This is done, in such a way that the first derivative of N respect to h is continuous everywhere, but not the second derivative. The error jumps at the points where the second derivative is discontinuous.

In Figure 10 the variation of ϵ is shown for a simple Chapman distribution of electron density. The corresponding errors are plotted versus height in Figures 10 and 11 for different values of $\Delta_1 r$. It can be seen that when $\Delta_1 r$ is relatively large, a systematic error is produced. Reducing the increment $\Delta_1 r$ the systematic error decreases to a point where random errors appear. A further decrease of $\Delta_1 r$ results in an increase of the random error. The error ϵ is not a function of $\Delta_1 r$ alone; but it depends on many additional

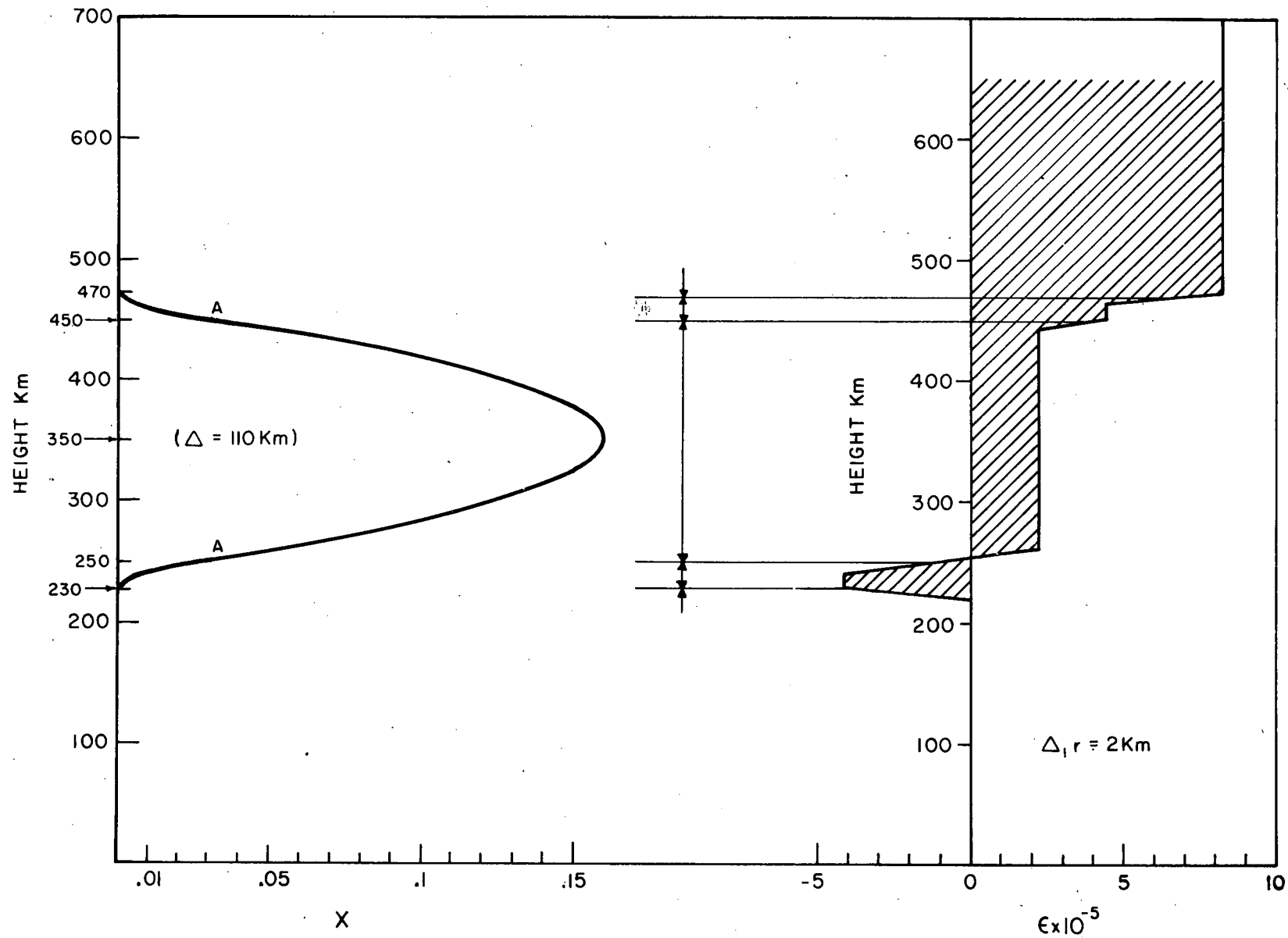


Figure 9. "Smooth" parabolic layer and ϵ as a function of height

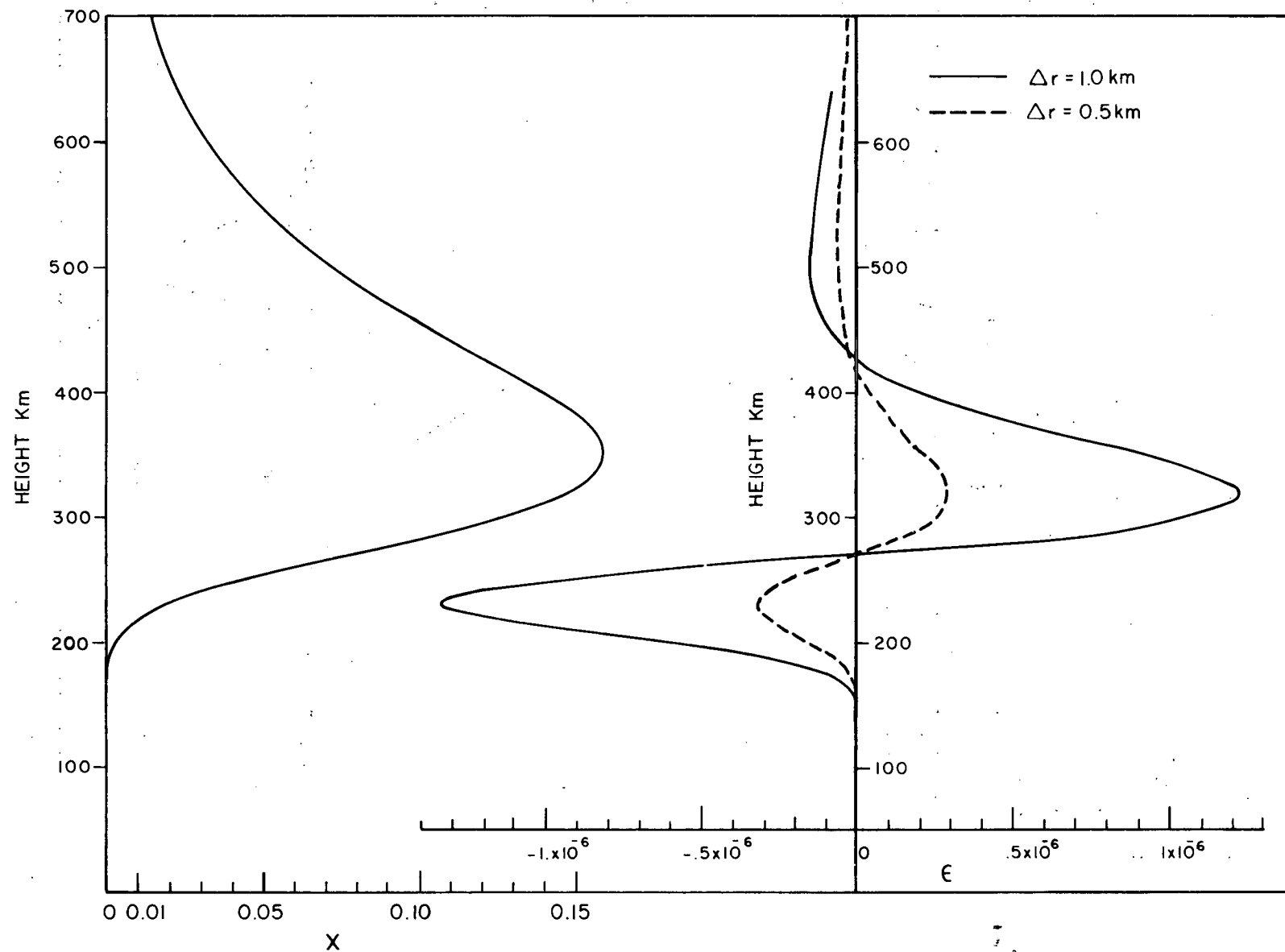


Figure 10. Chapman layer and ϵ as a function of height

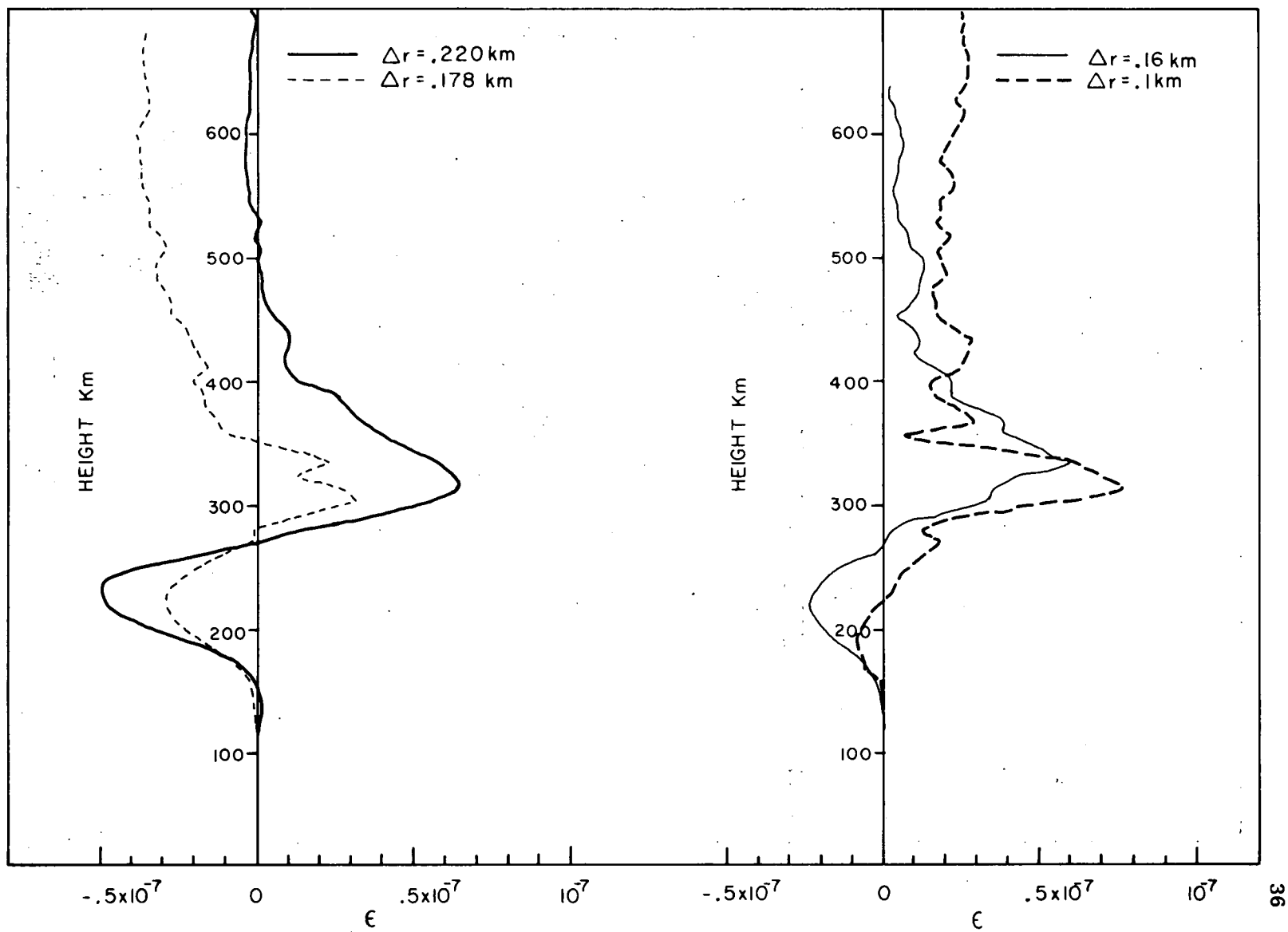


Figure 11. ϵ as a function of height

factors which have been kept constant when the curves of Figures 8, 10, and 11 were obtained.

The subroutine (EWW) is a straightforward computation of the values (6.1) with the help of the ray tracing equations.

7. SUBROUTINES USED

The theory and function of the principal subroutines have been explained in Sections 4 and 6. This section will be devoted to subroutines which have not been mentioned or which deserve special attention, specifically, the (NSQ), (ED), and (EMF) subroutines.

7.1 ILLIAC Library Subroutines:

- (R1) Square root - Given $x > 0$ it will calculate \sqrt{x} (Wheeler, 1958¹⁵)
- (S4) Exponential - Given $-1 < x < 0$ it will calculate e^x (Goldberg, 1956¹⁶)
- (T5) Sin-cosine - Given θ , it will calculate $1/2 \sin \theta$ (Wermer, 1958¹⁷)
- (P16) Infraprint - Given x it will print correctly rounded up to 12 figures in fractionary or integer form with the decimal point anywhere (Gillies, 1958¹⁸)
- (N12) Infraput - It will read sequence of numbers up to 12 figures in fractionary or integer form (Gillies, D. B., 1957¹⁹)
- (Y5) Transfer block of words between the Williams Memory and the Magnetic Drum (Elliott, 1960²⁰)

Additional Subroutines:

(SEC) Given $x \leq \frac{1}{2}$ it calculates $\sqrt{\frac{1}{4} - x^2}$, or given $1/2 \sin \theta$ it calculates $1/2 \cos \theta$.

(RP) Given $1/2 \sin \theta$ it calculates $-90^\circ \leq \theta \leq 90^\circ$

(SEC) and (RP) have standard entry.

7.2 Special Subroutines:

1) (NSQ) This program calculates the ordinary or extraordinary values of μ^2 and D or μ^2 only if the following data has been conveniently prepared: H , N , $\sin \Theta$, $\cos \Theta$, and f .

This subroutine has the following constants stored:

$$K_1 = \frac{e\mu_o}{2\pi m} \qquad K_o = \frac{e^2}{4\pi^2 m \epsilon_o} \qquad (7.1)$$

(NSQ) calculates in the first seven words the parameters

$$K_{11} = \frac{K_1}{f} \qquad K_{oo} = \frac{f^2}{K_o} \qquad (7.2)$$

and (NSQ) is arranged in such a way that, if the same frequency is used throughout a certain program (after the first entry to this subroutine) the next entries can be done to the right hand order of the word number seven of this subroutine thus saving computation time.

A shortened version of the program has been prepared for the case in which the magnetic field is disregarded. In this case Equation 3.1 reduces to:

$$\mu^2 = 1 - X \quad (7.3)$$

2) (EMF) Given r , θ and φ , this subroutine will calculate the earth's magnetic field H and its components referred to the local orthogonal system. Three models have been used:

a) Constant magnetic field

$$H_r = \text{const} \quad H_\theta = \text{const} \quad H_\varphi = \text{const} \quad (7.4)$$

b) Matched dipole approximation

$$\begin{aligned} H_r &= -2 \left(\frac{a}{r}\right)^3 [g_1^0 \cos \theta + \sin \theta (g_1^1 \cos \varphi + h_1^1 \sin \varphi)] \\ H_\theta &= \left(\frac{a}{r}\right)^3 [-g_1^0 \sin \theta + \cos \theta (g_1^1 \cos \varphi + h_1^1 \sin \varphi)] \\ H_\varphi &= \left(\frac{a}{r}\right)^3 [-g_1^1 \sin \varphi + h_1^1 \cos \varphi] \end{aligned} \quad (7.5)$$

The constants g_1^0 , g_1^1 and h_1^1 are to match the three components of the local magnetic field.

c) Complete field model (Jones and Melotte, 1953²¹)

$$\begin{aligned} H_r &= - \sum_n \sum_m \left(\frac{a}{r}\right)^{n+2} (n+1) H_n^m (g_n^m \cos m \varphi + h_n^m \sin m \varphi) \\ H_\theta &= -\text{cosec } \theta \sum_n \sum_m \left(\frac{a}{r}\right)^{n+2} H_n^m (-g_n^m \sin m \varphi + h_n^m \cos m \varphi) \end{aligned} \quad (7.6)$$

$$H_{\varphi} = -\operatorname{cosec} \theta \sum_n \sum_m \left(\frac{a}{r}\right)^{n+2} \sin \theta \frac{dH_n^m}{d\theta} (g_n^m \cos m \varphi + h_n^m \sin m \varphi) \quad (7.6)$$

Where:

$$H_n^m(\cos \theta) = \frac{2^h n! (n-m)!}{(2-n)!} P_{mn}(\cos \theta) \quad (7.7)$$

P_{mn} = associate Legendre polynomial

g_n^m, h_n^m = coefficients with the dimension of amper per meter. Those calculated by Finch and Leaton (1957²²) are tabulated in Appendix I.

3) (ED) This program calculates the electron density as a function of the coordinates (r, θ, φ)

a) Simple parabolic layer

$$N = N_m \left[1 - \frac{(r - r_m)^2}{\Delta^2} \right] \quad \text{for } |r - r_m| < \Delta$$

$$N = 0 \quad \text{for } |r - r_m| > \Delta \quad (7.8)$$

where

N_m = maximum electron density of the layer

r_m = value of r corresponding to maximum electron density

Δ = semi-thickness of the layer

b) "Smooth" parabolic layer

$$N = N_m \left[1 - \frac{(r - r_m)^2}{ca} \right] \quad \text{for } |r - r_m| \leq c$$

$$N = N_m \frac{(a - |r - r_m|)^2}{a(a - c)} \quad \text{for } c \leq |r - r_m| \leq a \quad (7.9)$$

$$N = 0 \quad \text{for } |r - r_m| \geq a$$

where

a = semi-thickness of "smooth" parabolic layer

$\Delta = \sqrt{ac}$ = semi-thickness of equivalent simple parabolic layer

c) Simple Chapman layer

$$N = N_m e^{\frac{H - (r - r_m) - He \frac{-(r - r_m)}{H}}{2H}} \quad (7.10)$$

where

H = scale height of the layer

d) Chapman layer with a gradient along meridian

$$N = N_m e^{\frac{H - (r - r_m) - He \frac{-(r - r_m)}{H}}{2H}} \quad (7.11)$$

$$N_m = \frac{N_{m,i+1} \cdot (\theta - \theta_i) + N_{m,i} \cdot (\theta_{i+1} - \theta)}{\theta_{i+1} - \theta_i}$$

$$r_m = \frac{r_{m,i+1} \cdot (\theta - \theta_i) + r_{m,i} \cdot (\theta_{i+1} - \theta)}{\theta_{i+1} - \theta_i}$$

$$H = \frac{H_{i+1} \cdot (\theta - \theta_i) + H_i \cdot (\theta_{i+1} - \theta)}{\theta_{i+1} - \theta_i}$$

where

$$\theta_{i+1} > \theta_i \quad \text{for } i = 0, 1, 2, 3, 4, 5, 6, 7$$

The values N_{mi} , r_{mi} , H_i corresponding to the values θ_i are stored in the program.

8. SCALING AND LIMITATIONS

The ray tracing differential equations have the following limitations:

- a) Limitations inherent in the ray theory discussed in Section 2.
- b) Limitations in the value of θ mentioned in Section 5.
- c) Limitations to be explained in this section concerning the range of values that the intervening variables are allowed to take.

These additional limitations, arise from the following two facts:

- (1) The computer will not operate with numbers equal to or larger than one.
- (2) The necessary scaling of the different variables in a convenient way so that they will accommodate the requirements of the computer.

The scaling of the variables is difficult because they have a very large range of variation. Another difficulty arises because accuracy is lost when full use is not made of the available digits in the computer. A compromise has been made in this particular program that keeps in mind its application to the use of data collected from artificial satellites. A floating point arithmetic has not been used because it would require exceedingly long computer time. This makes its use prohibitive.

The scaling of the principal variables is shown in the following list; where the primed variables are the scaled ones:

$$r' = r \text{ (Km)} \times 10^{-5}$$

$$t' = t \text{ (Km)} \times 10^{-5}$$

$$\alpha' = \frac{\alpha \text{ (radians)}}{\pi}$$

$$= \frac{\alpha \text{ (degrees)}}{180}$$

α is any angle

(8.1)

$$d' = d \text{ (Km)} \times 10^{-5}$$

d is any distance

$$v^i' = v^i \times 2^{-2}$$

$i = r, \theta, \varphi$

$$\mu' = \mu \times 2^{-2}$$

$$\mu'' = \mu \times 2^{-\frac{1}{2}}$$

two scalings are used

$$\sin \alpha = \sin \alpha \times 2^{-1}$$

α is any angle

$$D' = D \times 2^{-1}$$

$$N' = N \text{ (electrons per cubic meter)} \times 10^{-14}$$

(8.1)

$$H' = H \text{ (ampers per meter)} \times 10^{-2}$$

$$f' = f \text{ (cycles per second)} \times 2^{-39}$$

In addition to the direct scaling of the variables, the scaling of simple functions should be considered in the Auxiliar Subroutine and in (NSQ).

The following simple functions are defined in the auxiliar subroutine:

$$E_r = \frac{1}{8\mu'^2} [v^{r'} - 4D' (v^{r'} \cos' \Theta - \mu' \cos' \eta_r)]$$

$$E_\theta = \frac{1}{8\mu'^2} [v^{\theta'} - 4D' (v^{\theta'} \cos \Theta - \mu' \cos' \eta_\theta)]$$

$$E_\varphi = \frac{1}{8\mu'^2} [v^{\varphi'} - 4D' (v^{\varphi'} \cos \Theta - \mu' \cos' \eta_\varphi)]$$

$$K_\theta = \frac{2v^{\theta'}}{r'}$$

$$K_\varphi = \frac{2v^{\varphi'}}{r'}$$

(8.2)

$$\beta = \frac{\cos' \theta}{\sin' \theta}$$

$$\Delta_r \mu''^2 = \mu''^2 (r + \frac{1}{2} \Delta_1 r, \theta, \varphi, v^r, v^\theta, v^\varphi) - \mu''^2 (r - \frac{1}{2} \Delta_1 r, \theta, \varphi, v^r, v^\theta, v^\varphi)$$

$$\Delta_\theta \mu''^2 = \mu''^2 (r, \theta + \frac{1}{2} \Delta_1 \theta, \varphi, v^r, v^\theta, v^\varphi) - \mu''^2 (r, \theta - \frac{1}{2} \Delta_1 \theta, \varphi, v^r, v^\theta, v^\varphi)$$

$$\Delta_\varphi \mu''^2 = \mu''^2 (r, \theta, \varphi + \frac{1}{2} \Delta_1 \varphi, v^r, v^\theta, v^\varphi) - \mu''^2 (r, \theta, \varphi - \frac{1}{2} \Delta_1 \varphi, v^r, v^\theta, v^\varphi)$$

$$\begin{aligned}
A_r &= 8 \Delta_1 r' \mu''^2 \\
A_\theta &= 8\pi \mu''^2 r' \Delta_1 \theta' \\
A_\varphi &= 16\pi \mu''^2 r' \sin' \theta \Delta_1 \varphi' \\
\Delta_m t &= 2^m \Delta t'
\end{aligned} \tag{8.2}$$

The values (6.1) obtained from the ray tracing equations can be written with the new scaled variables in the following way:

$$\begin{aligned}
2^m \Delta t' \frac{dr'}{dt'} &= 2 E_r \Delta_m t \\
2^m \Delta t' \frac{d\theta'}{dt'} &= \frac{2}{r' \pi} E_\theta \Delta_m t \\
2^m \Delta t' \frac{d\varphi'}{dt'} &= \frac{1}{r' \pi \sin' \theta} E_\varphi \Delta_m t \\
2^m \Delta t' \frac{dv^{r'}}{dt'} &= \left(\frac{\Delta_r \mu''^2}{A_r} + K_\theta E_\theta + K_\varphi E_\varphi \right) \Delta_m t \\
2^m \Delta t' \frac{dv^{\theta'}}{dt'} &= \left(\frac{\Delta_\theta \mu''^2}{A_\theta} - K_\theta E_r + \beta K_\varphi E_\varphi \right) \Delta_m t \\
2^m \Delta t' \frac{dv^\varphi}{dt'} &= \left(\frac{\Delta_\varphi \mu''^2}{A_\varphi} - K_\varphi E_r - \beta K_\theta E_\theta \right) \Delta_m t
\end{aligned} \tag{8.3}$$

Each of the functions defined in (8.2) is independently scaled in different parts of the program. Each of them also has its own limitation of not exceeding the value one.

$$\begin{aligned}
E'_r &= E_r \\
E'_\theta &= E_\theta \\
E'_\varphi &= E_\varphi
\end{aligned} \tag{8.4}$$

$$K'_{\theta} = K_{\theta} \times 2^{-4}$$

$$K'_{\phi} = K_{\phi} \times 2^{-4}$$

$$\beta' = \beta \times 2^{-4}$$

$$\Delta'_r \mu^2 = \Delta_r \mu''^2 \times 2^7$$

$$\Delta'_{\theta} \mu^2 = \Delta_{\theta} \mu''^2 \times 2^7$$

$$\Delta'_{\phi} \mu^2 = \Delta_{\phi} \mu''^2 \times 2^7$$

(8.4)

$$A'_r = A_r \times 2^{16}$$

$$A'_{\theta} = A_{\theta} \times 2^{16}$$

$$A'_{\phi} = A_{\phi} \times 2^{16}$$

$$\Delta'_m t = \Delta_m t \times 2^9$$

The scaling of the simple functions defined in (7.2) which occur in (NSQ) are:

$$K'_{oo} = K_{oo} 2^{-3} \times 10^{-14}$$

(8.5)

$$K'_{11} = K_{11} \times 10^2$$

The range of values that the different variables and other intervening quantities can take in an application of the program can be deduced from the scaling (8.1), (8.4) and (8.5). In each case, the most severe limitations should be considered.

The only exception which can be listed, of a variable exceeding the value one without affecting the end result, is that of scaled angles. When the value of a scaled angle exceeds the value one, it takes the value of the equivalent angle with the sign changed. The following example of the ILLIAC arithmetic explains this feature:

$$175^{\circ} + 20^{\circ} = -165^{\circ} \quad (8.6)$$

The range of allowed values of each variable will be examined in detail.

Electron Density - From the tenth equation of (8.1)

$$N < 10^{14} \quad (8.7)$$

Magnetic Field - From the eleventh equation of (8.1)

$$H < 100 \text{ Amperes per meter} \quad (8.8)$$

The scaling of the electron density and magnetic field satisfies the extreme conditions which occur in the ionosphere and the earth's magnetic field.

Frequency - From (8.5)

$$3.52 \text{ mc} < f < 254 \text{ mc} \quad (8.9)$$

Refractive Index - From the seven scaling of (8.1) and the first three of (8.4)

$$\frac{1}{4} < \mu^2 < 2 \quad (8.10)$$

The maximum limit is 2, however, the region of interest is:

$$\frac{1}{4} < \mu^2 \leq 1 \quad (8.11)$$

When Y is small, (8.11) corresponds approximately to:

$$\frac{3}{4} > x \geq 0 \quad (8.12)$$

The values v^r , v^{θ} and v^{φ} are components of a vector of magnitude μ , and the only limitation is:

$$v^i < 2 \quad \text{for } i = r, \theta, \varphi$$

Finite Differences $\Delta_1 r$, $\Delta_1 \theta$ and $\Delta_1 \varphi$

These values are constants stored in the subroutine (DER), and could be changed if desired, providing that the limitations (8.1) are considered; otherwise, the corresponding scaling should be changed. From the seventh, eighth, and ninth of (8.4):

$$\Delta_1 r < .3814 \text{ Km}$$

$$\Delta_1 \theta < \frac{1}{r} \cdot 3814 \text{ radians} \quad (8.14)$$

$$\Delta_1 \varphi < \frac{1}{r} \cdot 3814 \text{ radians}$$

The stored values are:

$$\Delta_1 r = .22 \text{ Km}$$

$$\Delta_1 \theta = 1.5708 \times 10^{-5} \text{ radians} \quad (8.15)$$

$$\Delta_1 \varphi = 1.5708 \times 10^{-5} \text{ radians}$$

Partial Derivatives of Refractive Index

Two limitations apply to the derivatives: The first one concerns the finite differences of the refractive index and their own scaling given in (8.4):

$$\Delta_r \mu^2 < 2^{-6}$$

$$\Delta_\theta \mu^2 < 2^{-6} \quad (8.16)$$

$$\Delta_\varphi \mu^2 < 2^{-6}$$

The other, and more significative, limitation arises from the ratio of the scaled values which appear in the first term of the right hand of the last three Equations (8.3):

$$\frac{\partial(\mu^2)}{\partial r} < \mu^2 \times 2^{12} \times 10^{-5} \text{ Km}^{-1}$$

$$\frac{\partial(\mu^2)}{\partial \theta} < r \mu^2 \times 2^{12} \times 10^{-5} \quad (8.17)$$

$$\frac{\partial(\mu^2)}{\partial \varphi} < r \sin \theta \mu^2 2^{12} \times 10^{-5}$$

Distances

In general, the distances have the only limitation which arises from the corresponding scaling:

$$d < 100,000 \text{ Km} \quad (8.18)$$

Coordinates

The limitations for r are from (8.14), (8.15) and the fifth (8.4):

$$5,250 \text{ Km} < r < 24,280 \text{ Km} \quad (8.19)$$

r could take values larger than the one mentioned if the values (8.15) were chosen differently.

The limitations for θ come from the scaling of β given in the sixth of (8.4)

$$3^\circ 34' 35'' < \theta < 176^\circ 25' 25'' \quad (8.20)$$

No limitation exists for φ due to the arithmetic of the ILLIAC for the angles shown in (8.6)

Variable t - Its limitation arises from the scaling given in (8.1)

$$t < 100,000 \text{ Km} \quad (8.21)$$

Integral Step Length - From the last of (8.4)

$$2^m \Delta t \leq 2^{-9} 10^5 = 160 \text{ Km} \quad (8.22)$$

Since $m \geq 1$, the maximum integral step length is:

$$\Delta t = 80 \text{ Km} \quad (8.23)$$

9. PREPARATION OF DATA TAPE AND USE OF PROGRAM

A data tape is prepared for each particular program. The flow diagram of Figure 5 shows clearly the different part of the data tape and the need for a heading order for each group of data. Each group of data will be examined in detail in the first part of this section.

9.1 Frequency Data

26 217 N	Heading Order
f	Frequency in cycles in integer form
J	Terminating symbol which can be either of the following letters: N, J, F or L.

9.2 Electron Density Data

These data are prepared differently for different electron models. A simple spherically stratified Chapman layer is the only model used in the Faraday rotation program.

26 221 N	Heading order
H'	Scaled scale height in ($\text{Km} \times 10^{-5}$)
r' _m	Scaled radius corresponding to the point of maximum electron density ($\text{Km} \times 10^{-5}$)
N' _m	Scaled maximum electron density of the layer (electrons per $\text{m}^3 \times 10^{-14}$)
J	Terminating symbol which can be either N, J, F or L.

9.3 Earth Magnetic Field Model

These data are prepared according to the model used. At present, two models have been used:

Complete Field Model

26 219 N	Heading order
h ₁ ^{1/}	49 scaled coefficients ($\text{amper per meter} \times \frac{4}{100}$)
h ₂ ^{1/}	
h ₆ ^{1/}	

$h_2^2 /$

.

.

.

.

.

 $h_6^6 /$
 $g_o^o /$

.

.

 $g_6^o /$

.

 $g_6^6 /$

J Terminating symbol, either N, J, F or L

Matched Dipole

26 219 N Heading order

$g_1^o /$ Scaled g_1^0 (amper per meter $\times \frac{2}{100}$)

$g_1^1 /$ Scaled g_1^1 (amper per meter $\times \frac{4}{100}$)

$h_1^1 /$ Scaled h_1^1 (amper per meter $\times \frac{4}{100}$)

J Terminating symbol, either N, J, for L

9.4. Ray Sought Data

This datum is formed by two parts: the fractionary numbers and the integer numbers:

26 227 N Heading order

Fractionary Data

r_o'
 θ_o'
 ϕ_o'

} Scaled coordinates of point T or transmitter.

r_1'
 θ_1'
 ϕ_1'

} Scaled coordinates of point R or receiver.

$2^m \Delta t'$ Scaled value of integral step.

J Terminating symbol, either N, J, F or L

Integer Data

m 2^m is the scaling factor used in subroutine (F5) and (F1)

a If $a = 0$, the program will use the subroutine (F5) only.
 If it is a positive number larger than one, the program will use the subroutine (F1) and will print the coordinates of points every "a" integral steps.

b The program will print $\frac{1}{2} \mu_A^2$ and $\frac{1}{2} \epsilon$ every b printed points, or every a.b integral steps. If $b = 0$, $\frac{1}{2} \mu_A^2$ and $\frac{1}{2} \epsilon$ will not be printed at all.

c If $c = 0$ the program will first integrate the ordinary ray.
 If $c = 1$ the ray will integrate the extraordinary mode directly.

J Terminating symbol, either N, J, F or L

All the numbers of the data, fractionary or integers, can be written to twelve figures. These numbers are preceded by the sign + or - . (Gillies, 1957)

The program has in itself values of data corresponding to frequency, electron density, and magnetic field models. The arbitrarily chosen written-in values are:

For frequency:	$f = .20$ mc
For electron density:	$H = 60$ km
(Simple Chapman layer only):	$r_m = 6721$ km
	$N = .0079$

For magnetic field: The coefficients are those given by Finch and Leaton (1957) when the complete model is used. The coefficients for the matched dipole model are those obtained by matching the field with the "complete model" at 300 km above the receiving station at the University of Illinois.

Use of the program is fairly well explained in Figure 5. Listed below is a summary of the steps to follow:

Step 1. Read the program in the usual fashion. I should stop in a 24 999 (24 3F7₁₆) order at the right hand order of location 87 (057₁₆).

Step 2. Set the data of frequency, magnetic field, and electron density in reader. This can follow any order. Move black switch once for each group of data. After reading each group of data, the program will stop in a 24 999 (24 3F7₁₆) order at locations 218 (0JK₁₆), 220 (0JN₁₆) and 222 (0JF₁₆) respectively. If it is desirable to use written-in values, the corresponding group of data should be omitted.

Step 3. Set the ray sought data in the reader and move the black switch up once again. This will cause the program to start working.

Step 4. If the data has been prepared to integrate the ordinary mode, the program will stop in the B.S.S. 24 185 (24059₁₆) at the right hand order of location 180 (0S4₁₆) when the ray is completed. Move the black switch up if the extraordinary ray is desired. Return to Step 1 if a different ray is desired.

Step 5. When the extraordinary ray has been completed, the program will stop with the B.S.S. order 24 180 (24 0S5) at the left hand of location 180 (0S4₁₆). The program is then ready to proceed with a new ray. Return to Step 2.

10. PRELIMINARY RESULTS

The analysis of a passage of the satellite 1958 Δ_2 recorded at the receiving station of the University of Illinois was undertaken for the purpose of studying the behavior of the program described in this report.

The passage recorded between 0900 and 0908 C.S.T. on January 11, 1960 was chosen in accordance with the following:

- a) It was a passage close to the receiving station. The point of closest approach corresponds to a ground distance of about 200 km.
- b) It was a low passage. The satellite was at or below the height of maximum electron density of the ionosphere.
- c) There was an ionogram sounding taken at the University of Illinois at 0900 C.S.T.

The coordinates of the satellite were taken from the orbital data provided by the Smithsonian Astrophysical Observatory. These coordinates are shown in Figure 12. The points used for analysis correspond to times of integer minute and the points corresponding to time 0903, 0904, 0905, and 0906 will be called, Points 1, 2, 3, and 4 respectively. Figure 12 also shows the M.U.F. predictions published by the National Bureau of Standards (1959)²³ corresponding to the month of January.

The electron density distribution used was a spherically stratified Chapman layer (Equation 7.10). This layer is an approximation to the profile deduced from the mentioned ionogram. A comparison between both profiles is shown in Figure 13. The ionogram was reduced by the method described by Wright and Norton (1959)²⁴. The obtained parameters of the Chapman layer are:

$$H = 62.7 \text{ km}$$

$$h_M = 300 \text{ km}$$

$$N_M = 1.3416 \times 10^{12} \text{ elec per m}^3$$

The magnetic field used was a centered dipole (Equation 7.5) matched to the "complete field model" (Equation 7.6). This complete field mode uses the coefficients given by Finch and Leaton (1957) at a point 300 km above the

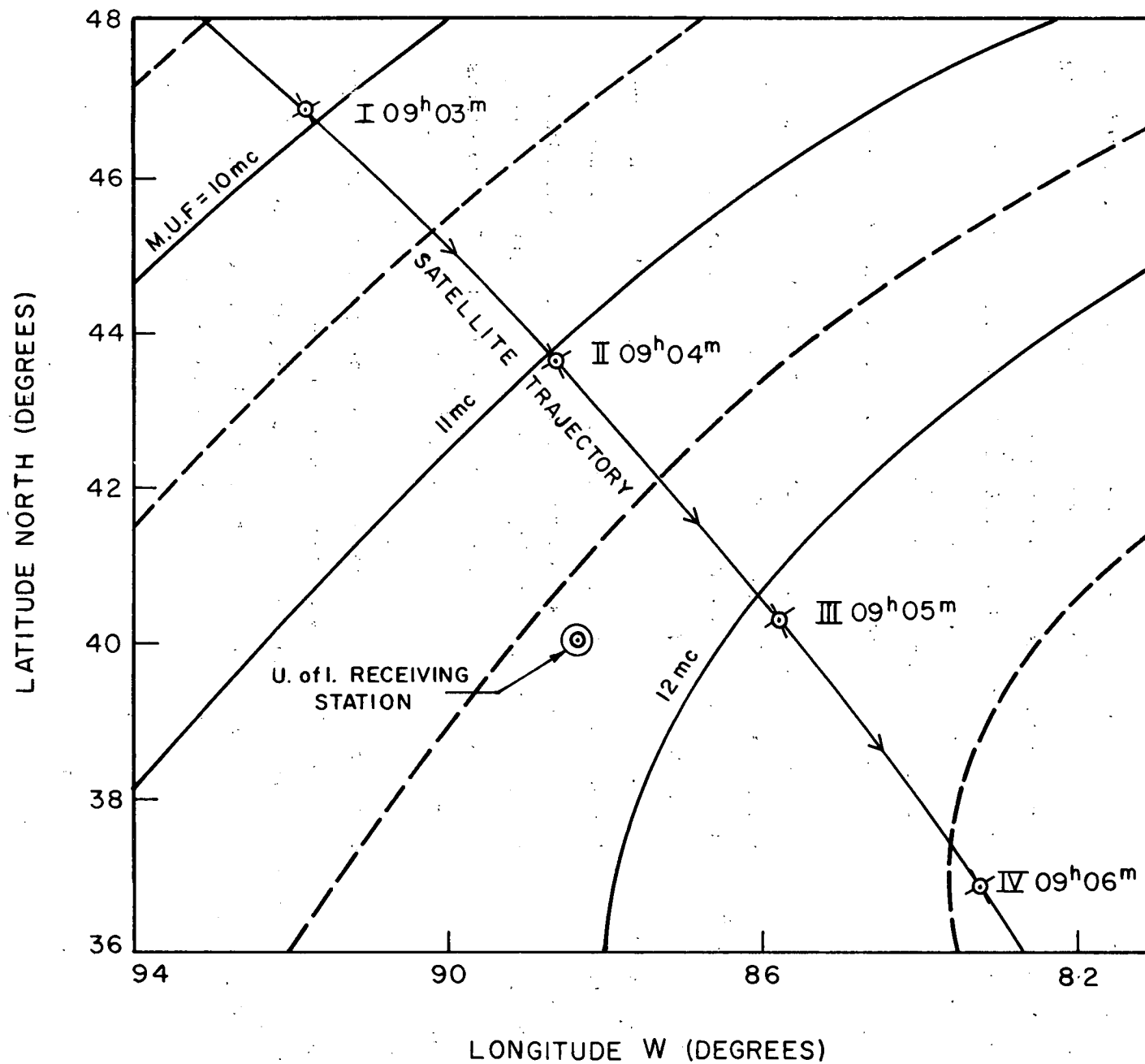


Figure 12. Trajectory of satellite and predictions of M.U.F.

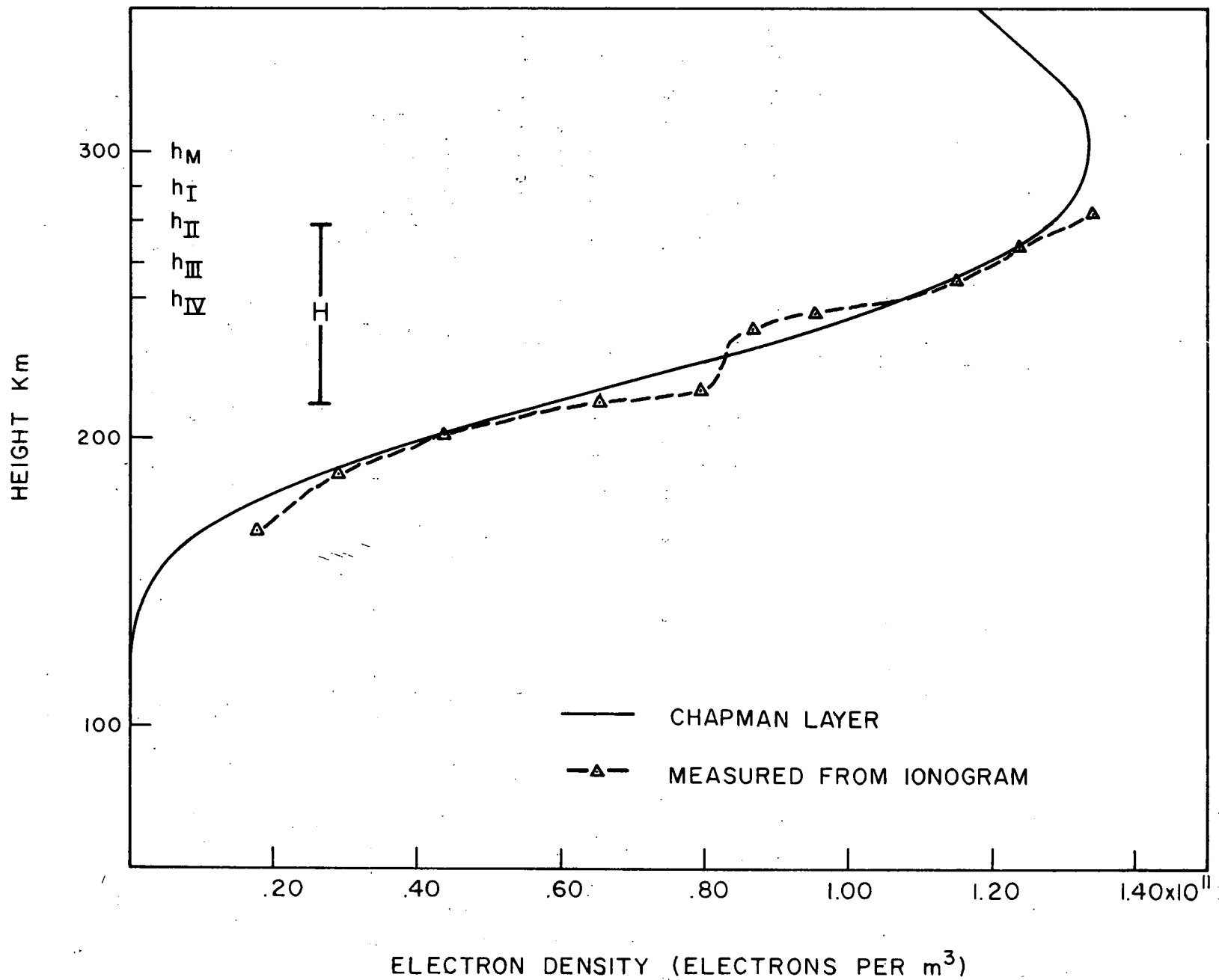


Figure 13. Assumed Chapman layer and measured electron density distribution as a function of height

receiving station. The values of the magnetic field at the matching point are:

$$H_r = -37.5208 \text{ Amp. per m}$$

$$H_\theta = -12.9927 \text{ Amp. per m}$$

$$H_\phi = + 1.0483 \text{ Amp. per m}$$

$$|H| = 39.7205 \text{ Amp. per m}$$

The ordinary mode rays obtained are shown in Figure 14. These rays have been plotted to gain an idea of the magnitude of the blending of the rays.

The lateral ground deviation of ordinary and extraordinary rays are shown in Figure 15.

The Faraday rotation was calculated for the four points with the values of integral step $\Delta t = 80 \text{ km}$, $\Delta t = 40 \text{ km}$ and for Point 3 with $\Delta t = 5 \text{ km}$. Table 1 shows the values of $t_{(o)}$, $t_{(x)}$ and Ω obtained in these calculations. It is interesting to notice that the resultant values of Ω are almost independent of the values of Δt , since the variations of $t_{(o)}$ and $t_{(x)}$ are in the same direction and compensate each other. This observation should be taken with reservations when very low rays are considered since in this condition a very small error in the components of $\bar{\mu}$ around the height of maximum electron density results in large difference on the path of the ray.

The r.m.s. values of ϵ , defined by Equation 6.3, have been calculated for the ordinary ray corresponding to these points. Together with $|\epsilon|_{\text{max}}$ these values are plotted as a function of phase length ($t_{(o)}$) in Figure 16. Contrary to expectations, the shortest ray, corresponding to Point 3, has the largest m.r.s. error. The same values of ϵ are plotted as a function of azimuthal angle, measured clockwise from the North, in Figure 17. The resulting curves of Figure 16 and 17 are smooth even though the sequence of points is different. This seems reasonable since the variation of the error ϵ depends on other factors such as electron density distribution, (as will be shown later), slope of the ray, and the angle between the direction of propagation and the magnetic field. It would be necessary to have a large number of systematic calculations to individualize the effect of each variable.

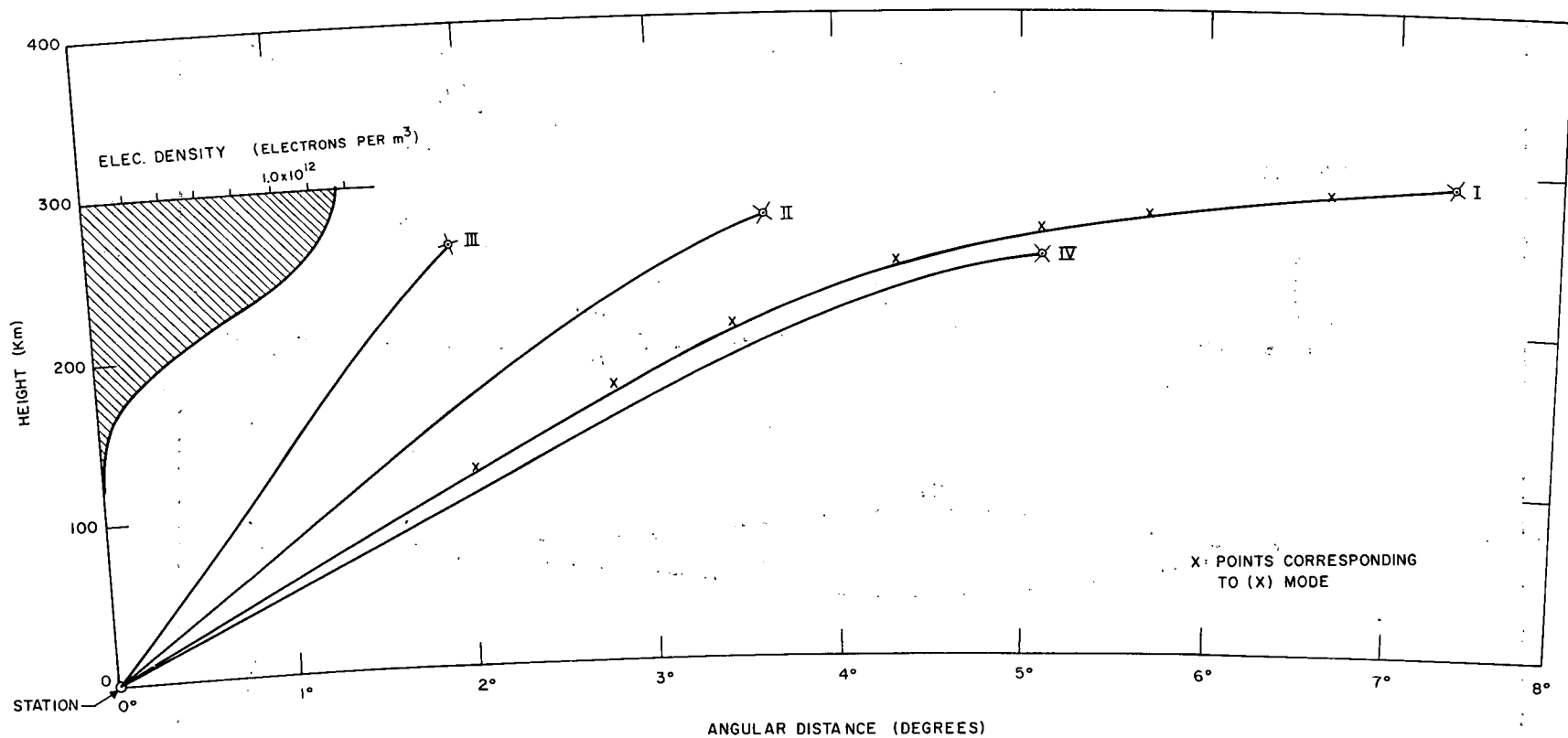


Figure 14. Ray trajectories and electron density distribution above station

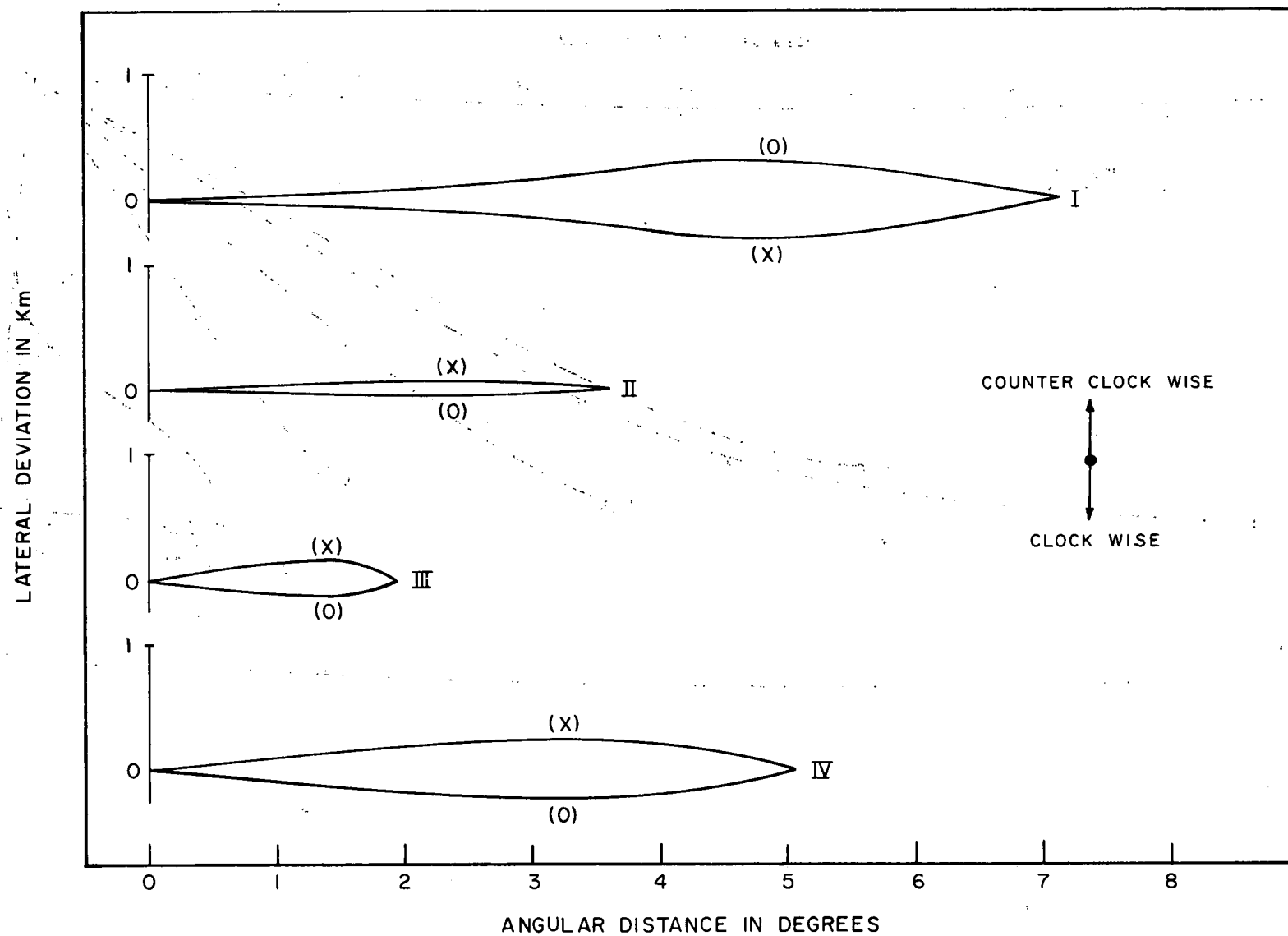


Figure 15. Horizontal ground deviation of ordinary and extraordinary rays

TABLE I

Coordinates of satellite 1958 $\Delta 2$ on January 11, 1960 and computed values of $t_{(o)}$, $t_{(x)}$ and Ω with different integral steps:

		Point 1	Point 2	Point 3	Point 4
	Time (C. S. T.)	09 hrs. 03 min.	09 hrs. 04 min.	09 hrs. 05 min.	09 hrs. 06 min.
	height (km)	291.7	276.6	262.3	249.0
	latitude (degrees)	46.88°N	43.63°N	40.27°N	36.82°N
	longitude (degrees)	91.97°W	88.62°W	85.79°W	83.24°W
$\Delta t = 80 \text{ km}$	$t_{(o)}$ (km)	839.617 440 2	479.752 390 3	334.788 727 9	617.033 499 8
	$t_{(x)}$ (km)	838.470 507 7	479.187 669 6	333.896 402 1	615.824 195 1
	Ω (rotations)	38.564 42	18.824 02	29.744 19	40.310 16
$\Delta t = 20 \text{ km}$	$t_{(o)}$ (km)	839.613 209 8	479.747 446 8	334.783 677 0	617.029 795 8
	$t_{(x)}$ (km)	838.462 668 4	479.178 110 8	333.889 749 5	615.819 564 0
	Ω (rotations)	38.351 35	18.977 9	29.797 58	40.341 06
$\Delta t = 5 \text{ km}$	$t_{(o)}$ (km)			334.783 018 3	
	$t_{(x)}$ (km)			333.889 701 1	
	Ω (rotations)			29.777 24	

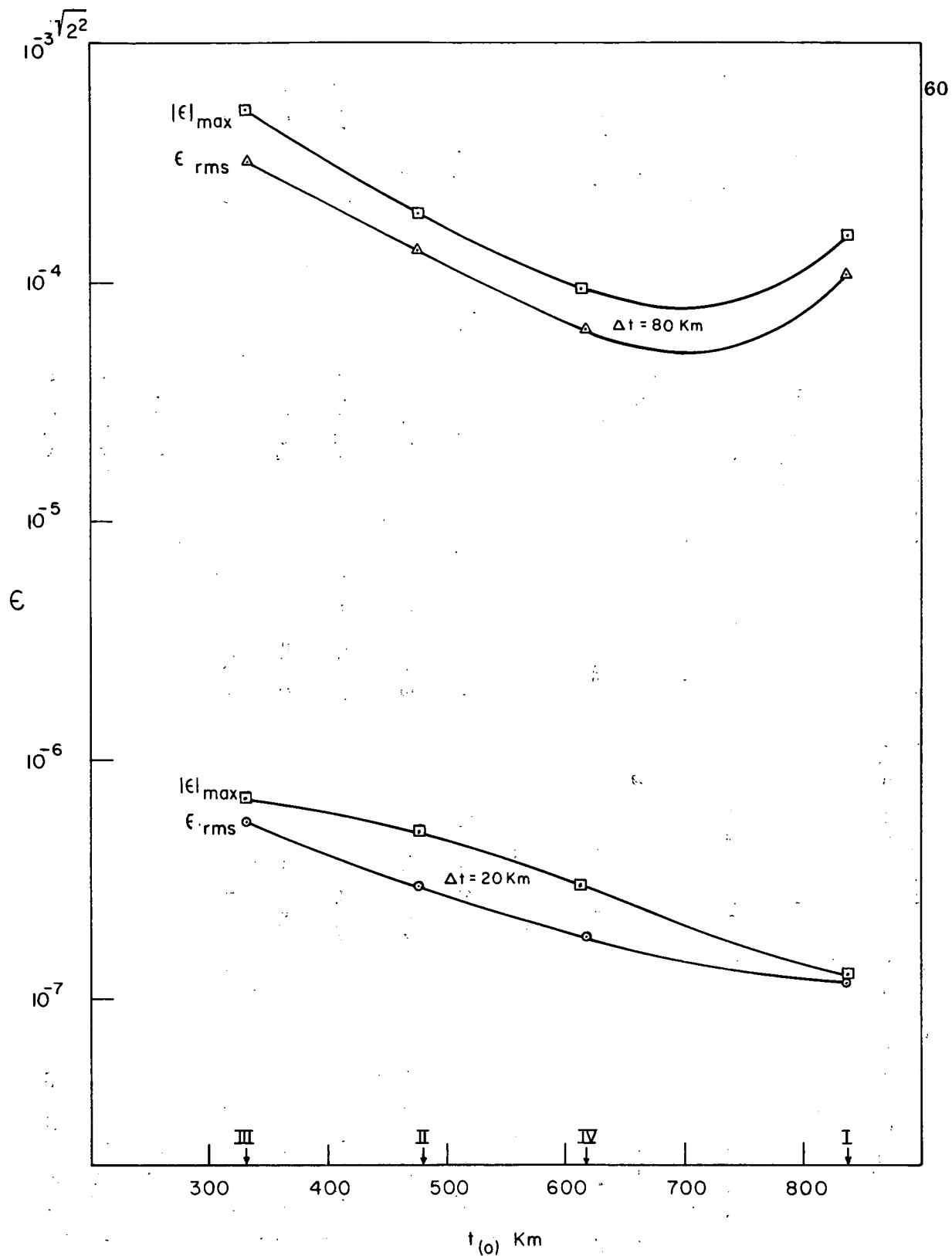


Figure 16. Variation of ϵ_{rms} and $|\epsilon|_{max}$ as a function of $t_{(o)}$

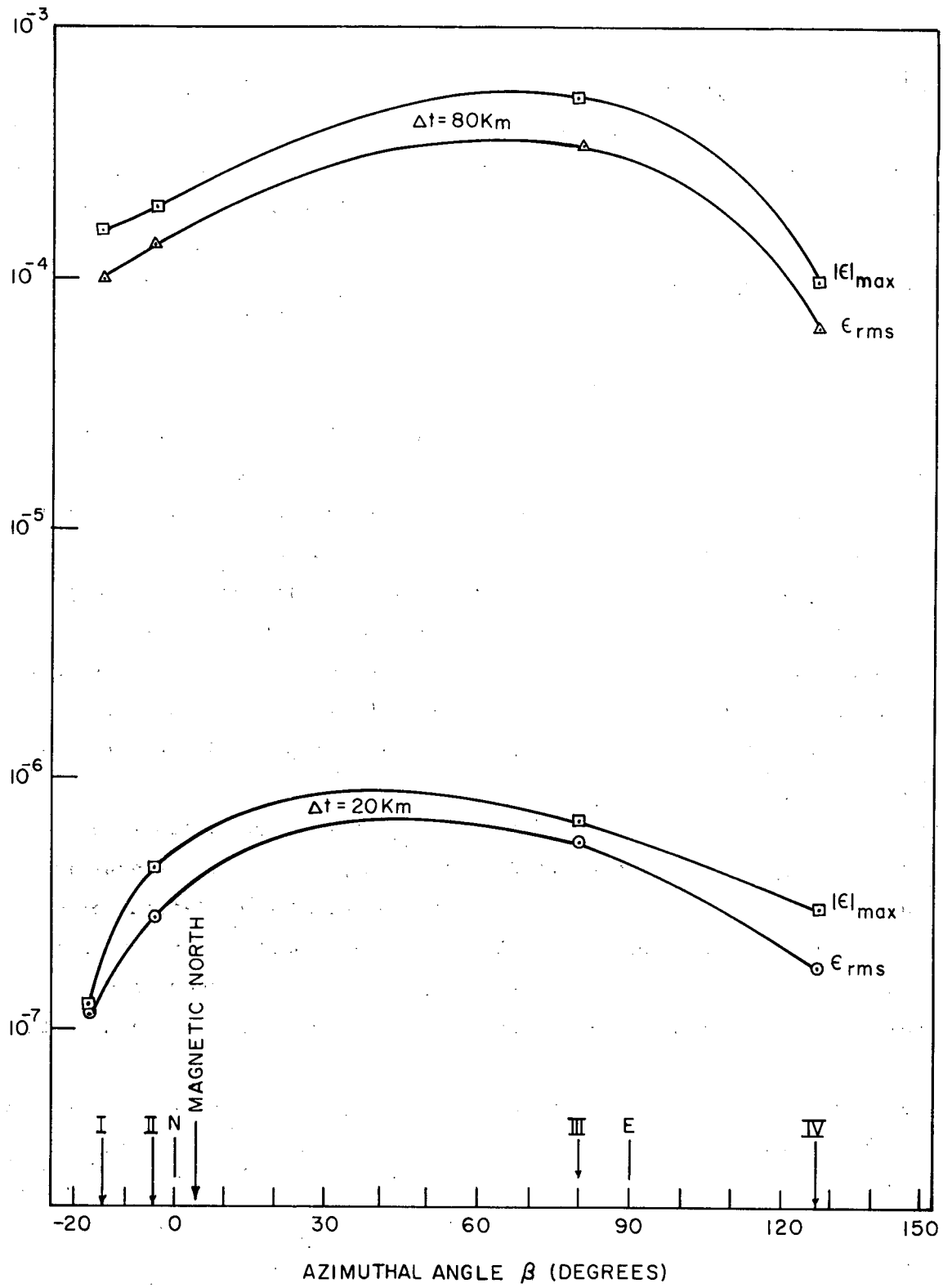


Figure 17. Variation of $\epsilon_{r.m.s.}$ and ϵ_{max} as a function of azimuthal angle β

The simplest example of individualizing the effect of one variable, and a very interesting one in the behavior of the program, is the following: The ordinary ray corresponding to Point 3 has been integrated using the values of Δt equal to 80, 40, 20, 10 and 5 km. The r.m.s. values of ϵ are plotted as a function of Δt in Figure 18. Note that the upper part of the curve between $\Delta t = 80$ km and $\Delta t = 20$ km follows the law:

$$\epsilon_{\text{r.m.s.}} = a (\Delta t)^b \quad (10.1)$$

where a and b are constants which can be calculated from Figure 18. The lower part of the curve decreases more slowly and it is expected that it will reverse its slope for smaller values of Δt , following a pattern very much like that explained in part 6, when discussing the variation of $\Delta_1 r$.

Decreasing the integral step Δt by the factor 1/2 increases the computation time by the factor 2. A value of $\Delta t = 20$ km is considered by the author a good compromise between computation time and obtained error. However, for rays not very critical, Δt could be increased to 80 km without appreciable difference in the final value of Faraday rotation. This has been pointed out with the aid of Table 1.

Another interesting feature is noticed by plotting ϵ as a function of height ($t_{(o)}$ or $t_{(x)}$ could also be used to this purpose) as it was done in Figure 19 for the Point 3 and for different values of Δt . Variations of ϵ occur only where there is appreciable electron density and remain constant below 80 km, where the electron density is negligible. This occurrence is more likely to be due to the presence of electron gradient rather than to the presence of the electron density alone.

Figure 20 is a comparison of computed and measured values of the "Faraday rotation", from the corresponding record obtained at the receiving station of the University of Illinois. The origin of the "Faraday rotation" axis for the measured curve is arbitrary. It was brought to coincide to the calculated one at the point of closest approach. The difference between the two curves is very large. However, an analysis of the assumptions used in the electron density and magnetic field models show that the difference is reasonable. The predictions of M.U.F. shown in Figure 12 indicate that the passage was at a time of the day

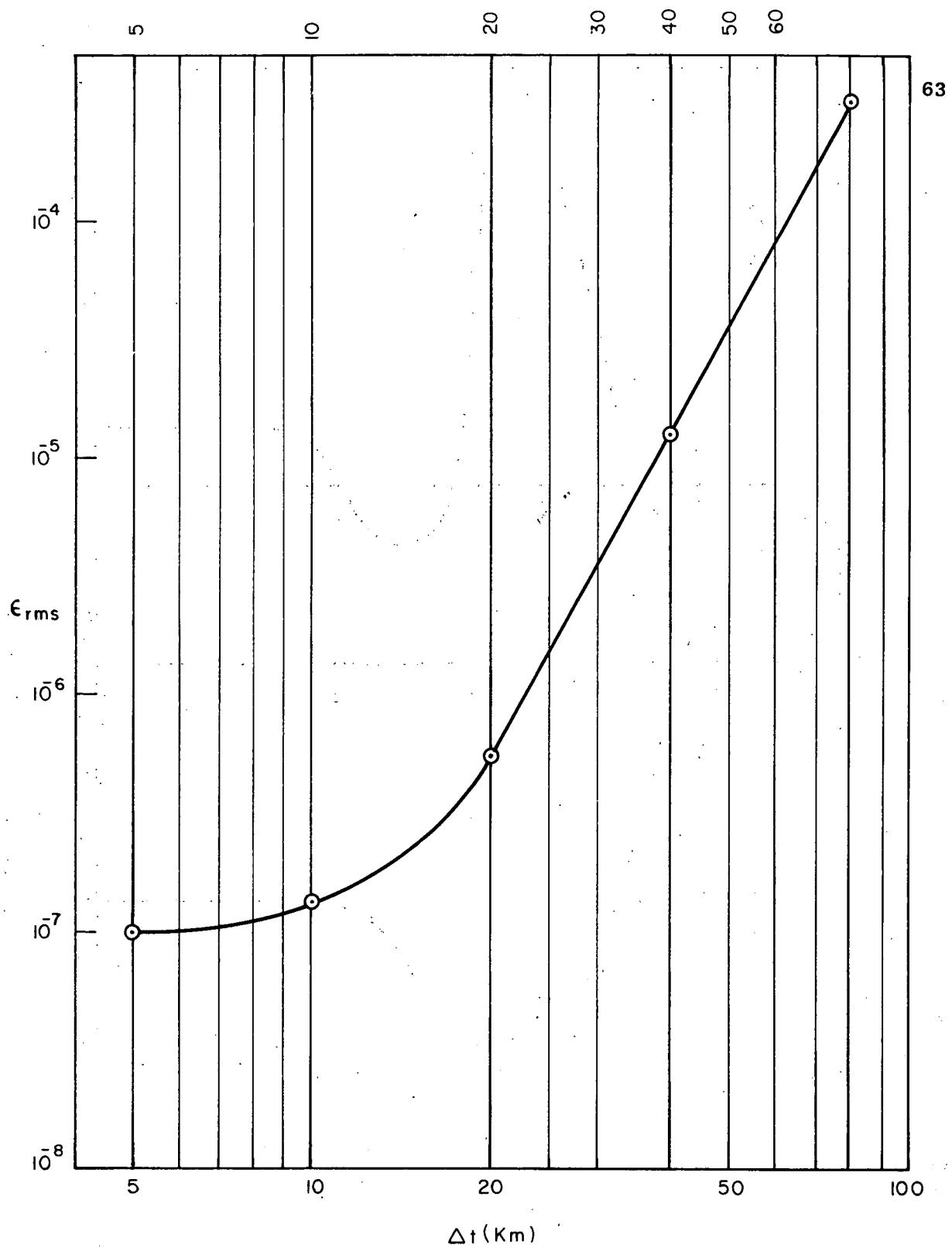


Figure 18. Variation of $\epsilon_{r.m.s.}$ as a function of Δt for point III

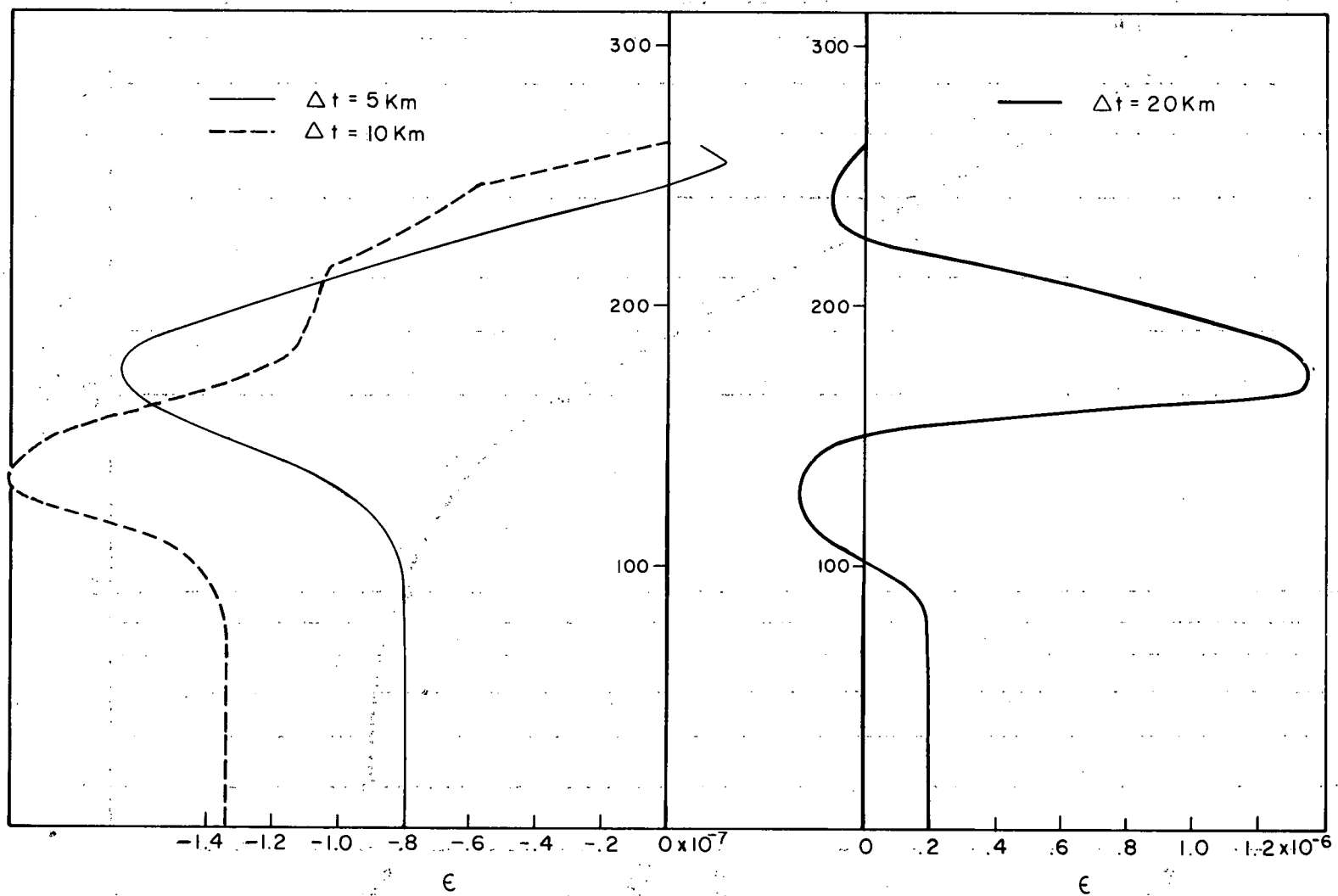


Figure 19. ϵ as a function of height for different values of Δt for point 3

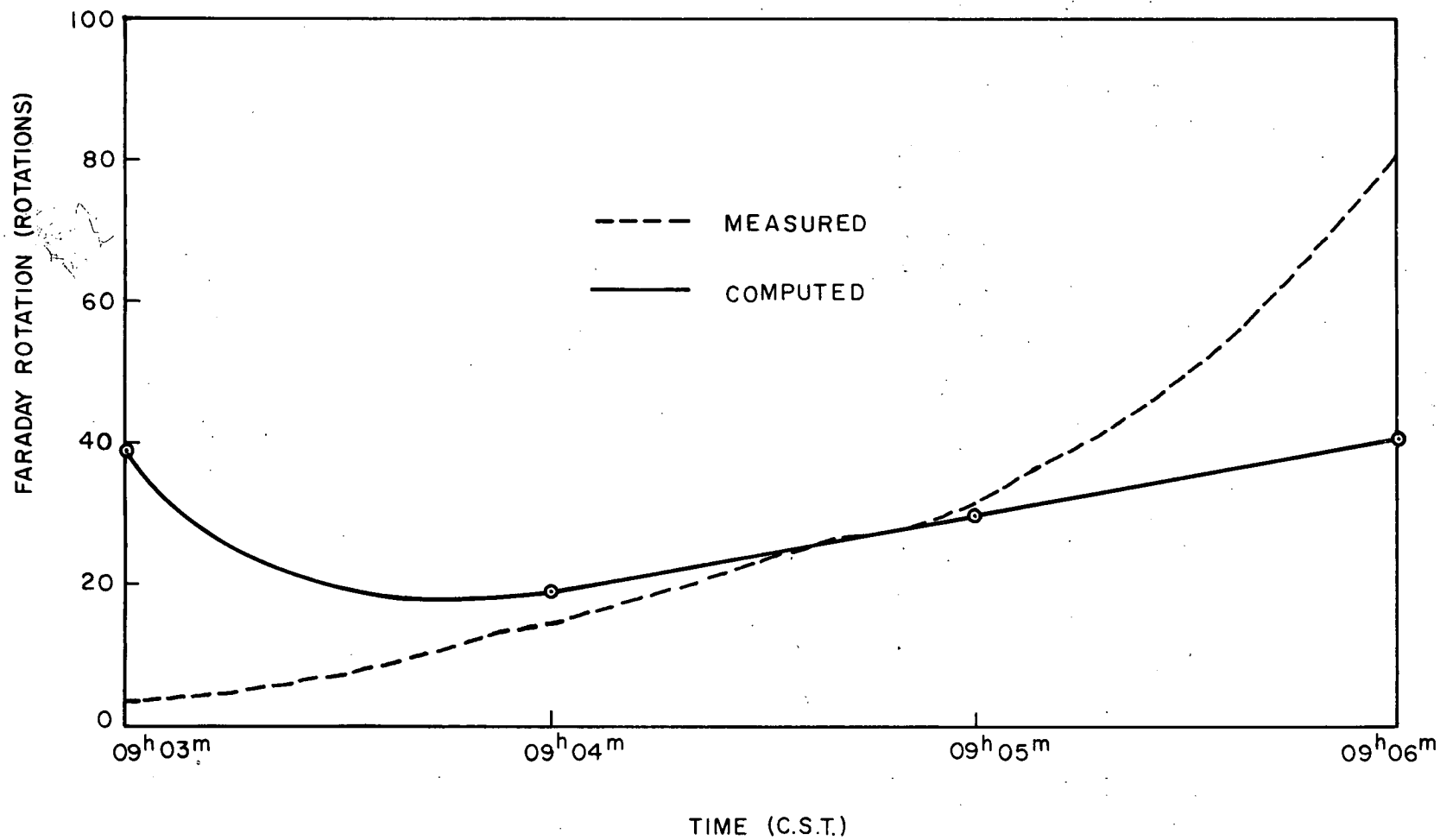


Figure 20. Computed and measured values of Faraday rotation as a function of time

when the horizontal gradient is very large. A smaller electron density at the N.W. would result in a smaller number of rotations at the beginning of the passage. A larger electron density at the S.E. would result in a larger number of rotations at the last part of the passage. In addition to this, it can be taken into account how critical the position of the satellite is to the electron density model used. The satellite was slightly below the peak of the ionosphere, and a small horizontal variation in the values of H and h_M , added to the variation of N_M could very deeply affect the final results.

The northern part of the passage is also very sensitive to the magnetic field model used as noted by Little, and Lawrence (1960).

From the foregoing discussion it can be seen how useful it would be to have an improved electron density model to arrive at a method of calculating the electron distribution from the Faraday rotation using artificial satellites.

11. PROPAGATION PROGRAM

A modified program, which considers a ray propagating in a meridian plane, has been prepared in order to investigate long range propagation problems. This program has the following characteristics:

1. The ray is contained in a meridian plane. The coordinate system used is a spherical one centered at the center of the Earth. Its orientation is determined for each problem so that the ray of interest is in a meridian plane.
2. The magnetic field is disregarded. Therefore, the refractive index is calculated with Expression (7.3).
3. The electron density distribution used is defined by Expression (7.11).
4. A downward going ray is mirror reflected when it reaches the surface of the Earth.
5. The output tape is suitable for use in the data plotter at the Digital Computer Laboratory of the University of Illinois (Carter, 1958²⁵).
6. The allowed ranges of variation for r and θ are given by (8.19) and (8.20).
7. The integration of the ray is interrupted either when r exceeds a given value r_F or when θ exceeds a given value θ_F , whichever happens first.

Data Tape This tape consists of two groups of data:

Electron Density Data This group of data consists of thirty-two decimal numbers up to twelve figures preceded by the sign plus:

$\theta'_0 - \theta'_0$	eight scaled angular distances $\theta'_i - \theta'_0$
$\theta'_1 - \theta'_0$	corresponding to points for which ionosphere data is available. θ is the initial value of θ for the ray sought.
$\theta'_2 - \theta'_0$	
.	
.	
$\theta'_7 - \theta'_0$	

H'_0	scaled height at the points
H'_1	defined by the previous set
H'_7	
N'_{M_0}	scaled maximum electron density at the same points
N'_{M_1}	
N'_{M_7}	
r'_{M_0}	scaled value of r at which the electron density is maximum
r'_{M_1}	at the same points.
r'_{M_7}	
N	Terminating symbol, either N, T, F or L

Ray Sought Data Consists of a set of fractionary numbers and a set of integer numbers. Each number is written up to twelve figures and preceded by a sign.

24 99 N

Heading order

Fractionary set of data

i_0

Scaled initial inclination angle. This angle determines the initial inclination of the ray, and it is measured from the up going verticle at the initial point.

θ'_o Scaled initial θ
 r'_o Scaled initial r
 θ'_f Scaled final θ
 r'_F Scaled final r
 $2^m \Delta t'$ Scaled integral step. It must satisfy (8.22)
 r'_{F5} Scaled r for (F5) test.
 v'^r_{F5} Scaled v^r_{F5} for (F5) test. r_{F5} and v^r_{F5} should be chosen so that a down coming ray that reaches the height corresponding to r_{F5} with a v^r smaller than v^r_{F5} will hit the surface of the Earth. The effect of the (F5) test is to transfer the integration to the subroutine (F5) which will integrate the ray exactly up to the surface of the Earth without any output in between. The ray is reflected at this point by reversing the sign of v^r . Suggested values for r_{F5} and v^r_{F5} are .643471 and -.0351 respectively.

N Terminating symbol. Either N, T, F, or L.

Integer set of data

a Output frequency. The program will print out the coordinates of a point every a integral steps.
 b Error output frequency. The program will print out $1/2 \mu_A^2$ and $1/2 \epsilon$ every a.b integral steps.
 m 2^m is the scaling factor used by subroutines (F1) and (F5).
 N Terminating symbol. Either N, T, F or L.

Use of Program The following steps should be observed when using the program.

Step 1. Read the program in the usual fashion. It will stop in a 24 999 (24 3F7₁₆).

Step 2. Move the black switch up once. This will cause the ILLIAC to read an "Auxiliar program" at the end of the tape. The program will stop in a 24 10 (24 K₁₆).

Step 3. Put the electron Density Data Tape in the reader and move the black switch up. When these data are read, the program will stop in a 24 99 (24 3F7₁₆) order at location 11 (S₁₆).

Step 4. Put the ray sought data tape in the reader and move the black switch up once. The program will start working. It will stop in a 34 999 (34 3F7₁₆) order at location 126(07F₁₆) or 128(080₁₆) when the ray is completed. At this point it is possible to go to step 2, or, if the same electron density data is to be used for a new ray, go back to Step 4.

Example. An application of this program is shown in Figure 21. The propagation of two rays of 15 mc between Slough, United Kingdom and San Francisco, California have been obtained using ionosphere data furnished by I.G.Y., U. S. World Data Center. The data are from stations near the meridian plane joining the two mentioned points and it corresponds to August 23, 1956, 00 hrs. 00 min. Universal time.

The largest sources of error in this application originate from the difficulty in analyzing the available ionograms and also in assuming that the stations from which ionosphere data was obtained were located on the meridian plane mentioned.

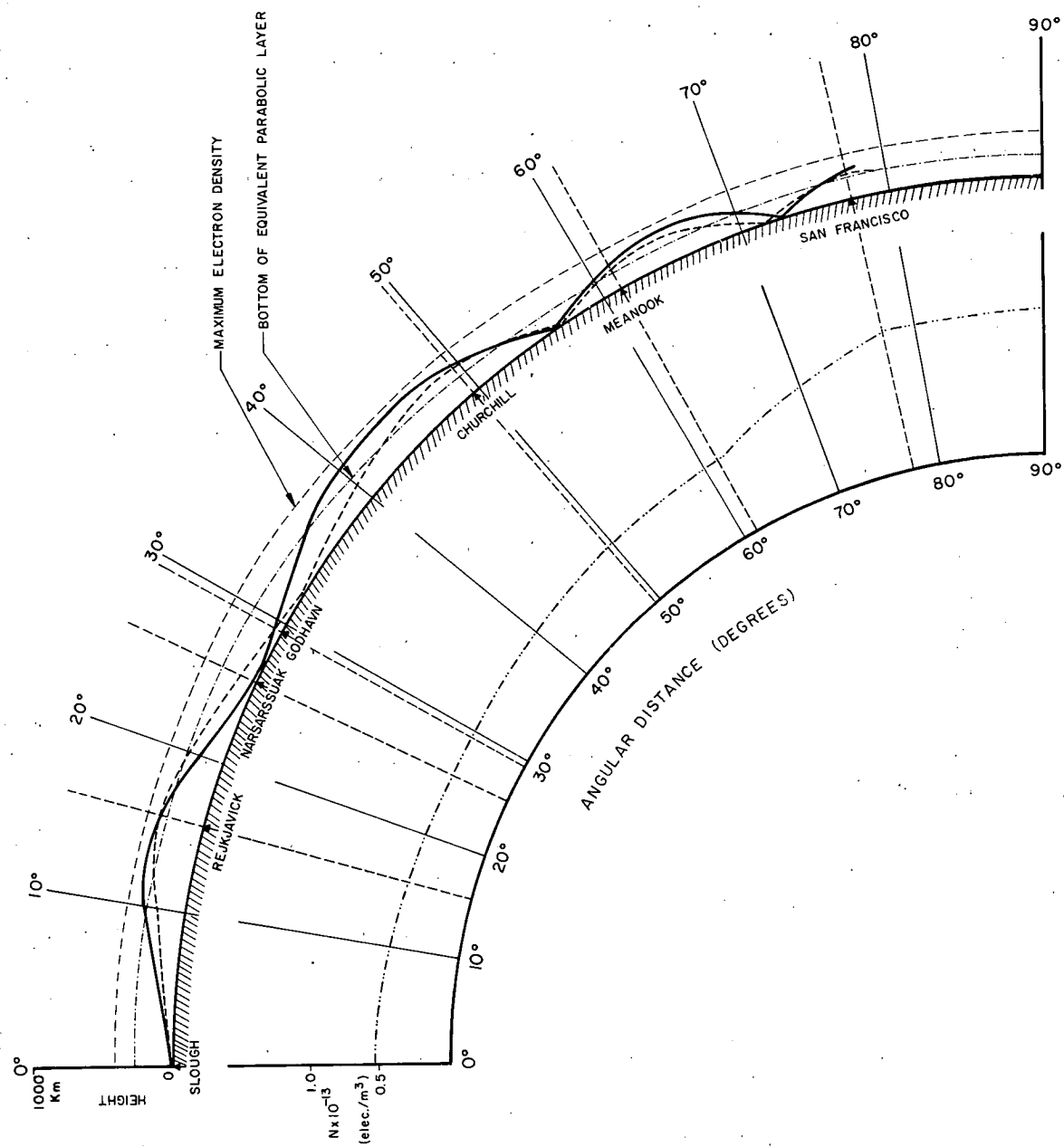


Figure 21. Propagation between slough (United Kingdom) and San Francisco (California) with a frequency of 15 mc.

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APPENDIX I

The constants used in the Faraday Rotation program are the following:

electron charge:

$$e = 1.6020 \times 10^{-19} \text{ coulombs (Cohen, 1955}^{26})$$

electron mass:

$$m = 9.10721 \times 10^{-31} \text{ kg (DuMond and Cohen, 1951}^{27})$$

velocity of light:

$$c = .2997925 \times 10^9 \text{ m sec}^{-1} \text{ (Froome, 1958}^{28})$$

magnetic permeability constant for free space:

$$\mu_0 = 4\pi \times 10^{-7} \text{ henry per meter}$$

dielectric constant for free space:

$$\epsilon_0 = \frac{1}{\mu_0 c^2} \text{ farads per meter}$$

Special constants for (NSQ) subroutine:

$$K_1 = \frac{e\mu_0}{2\pi m} = 35180.9170975 \text{ M.K.S. (r)}$$

$$K_2 = \frac{e^2}{4\pi^2 m \epsilon_0} = 80.6178714557 \text{ M.K.S. (r)}$$

Mean radius of the Earth:

$$a = 6371 \text{ km (Sitter, 1938}^{29})$$

Mathematical constants:

$$\pi = 3.14159265358979324$$

$$e = 2.718281828459045$$

Coefficients for "complete" earth magnetic field (Finch and Leaton, 1957)
are given in Table II

Location of receiving station of the University of Illinois:

lat: $40^{\circ} 01' 04.6''$ N

long: $88^{\circ} 19' 37.2''$ W

TABLE II

Gaussian coefficients in e.m.u system

 g_n^m

$n \quad m$	0	1	2	3	4	5	6
1	+.30545	+.02265					
2	+.02280	-.05245	-.01370				
3	-.02950	+.05855	-.02445	-.00720			
4	-.04170	-.04400	-.02260	+.00795	-.00230		
5	+.02035	-.03235	-.01550	+.00190	+.0034	+.00050	
6	-.01495	-.01005	-.00190	-.02350	+.00165	-.00010	+.00070

 h_n^m

$n \quad m$	0	1	2	3	4	5	6
1		-.05910					
2		+.03295	-.00215				
3		+.01390	-.00565	+.00080			
4		-.00810	+.01185	+.00080	+.00120		
5		-.00185	-.00715	+.00210	+.00315	-.00060	
6		+.00495	-.01715	.00000	+.00055	+.00060	+.00010

APPENDIX II

The purpose of this appendix is to sketch the transformation of Haselgrove's ray tracing equations from the rectangular coordinate system of reference R to a spherical system, S. The method will be a straightforward transformation of variables. The direct method has been chosen to avoid the use of tensorial calculus for the benefit of those who are not familiar with this branch of mathematics. In addition the transformation helps to increase familiarity with the meaning of the various derivatives and also proves how much work is saved when tensorial methods are used in the transformation of these equations.

The Haselgrove's ray tracing equations in a rectangular coordinate system are:

$$\frac{dx}{dt} = \frac{1}{\mu^2} (v^x - \mu \frac{\partial \mu}{\partial v^x})$$

$$\frac{dy}{dt} = \frac{1}{\mu^2} (v^y - \mu \frac{\partial \mu}{\partial v^y})$$

$$\frac{dz}{dt} = \frac{1}{\mu^2} (v^z - \mu \frac{\partial \mu}{\partial v^z})$$

(II-1)

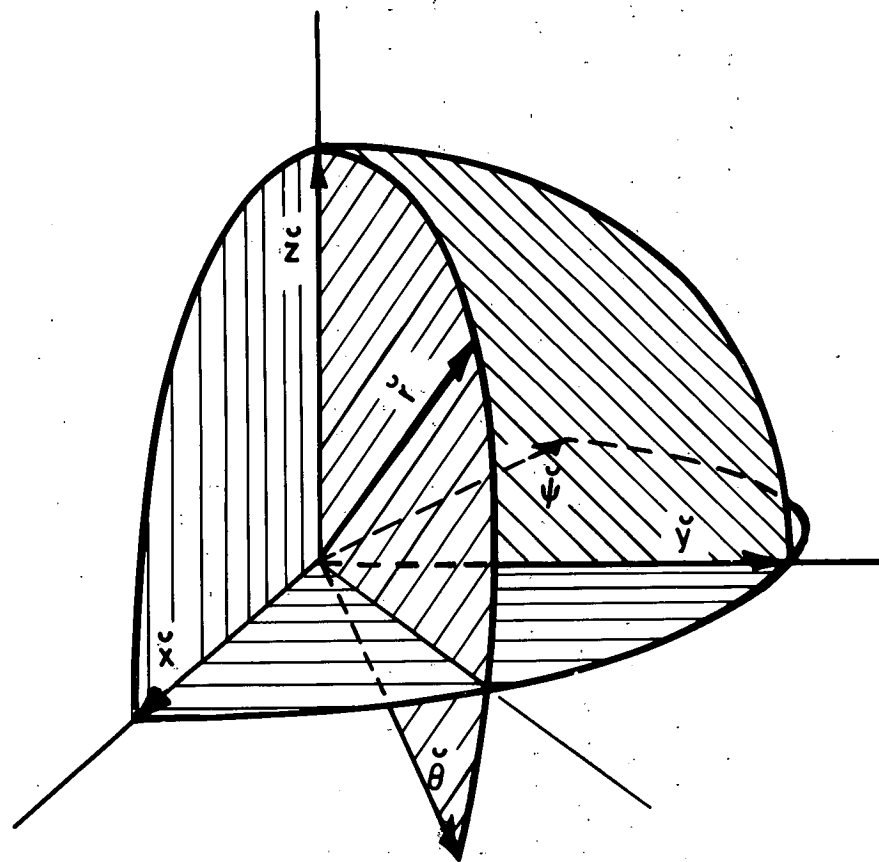
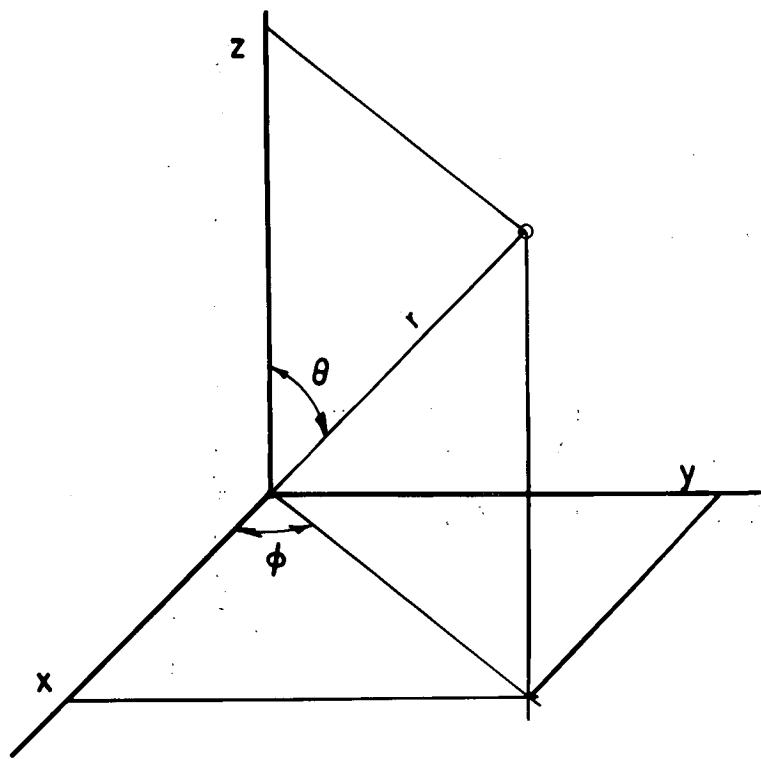
$$\frac{dv^x}{dt} = \frac{1}{\mu} \frac{\partial \mu}{\partial x}$$

$$\frac{dv^y}{dt} = \frac{1}{\mu} \frac{\partial \mu}{\partial y}$$

$$\frac{dv^z}{dt} = \frac{1}{\mu} \frac{\partial \mu}{\partial z}$$

From Figure 22, the variables of the S system can be written in function of the R system variables:

$$r = \sqrt{x^2 + y^2 + z^2} \quad (II-2)$$



$\hat{x}, \hat{y}, \hat{z}, \hat{r}, \hat{\theta}, \hat{\phi}$ ARE UNIT VECTORS

Figure 22. Relation between "R" and "S" variables

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}$$

$$\varphi = \tan^{-1} \frac{y}{x}$$

$$v^r = \frac{1}{\sqrt{x^2 + y^2 + z^2}} (v^x x + v^y y + v^z z) \quad (\text{II-2})$$

$$v^\theta = \frac{1}{\sqrt{x^2 + y^2 + z^2} \sqrt{x^2 + y^2}} [v^x xz + v^y yz - v^z (x^2 + y^2)]$$

$$v^\varphi = \frac{1}{\sqrt{x^2 + y^2}} (-v^x y + v^y x)$$

The inverse transformation will be:

$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \theta$$

(II-3)

$$v^x = v^r \sin \theta \cos \varphi + v^\theta \cos \theta \cos \varphi - v^\varphi \sin \varphi$$

$$v^y = v^r \sin \theta \sin \varphi + v^\theta \cos \theta \sin \varphi + v^\varphi \cos \varphi$$

$$v^z = v^r \cos \theta - v^\theta \sin \theta$$

Taking the total derivatives of (II-2) with respect to t , the result will be the sought derivatives of the variables of the S system in terms of the derivative of the R system in respect to time. Substituting the derivatives of the R variables by the values (II-1) and the R variables by the values (II-3), the resulting equations are:

$$\frac{dr}{dt} = \frac{v^r}{\mu^2} - \frac{1}{\mu} \left[\sin \theta \cos \varphi \frac{\partial \mu}{\partial v^x} + \sin \theta \sin \varphi \frac{\partial \mu}{\partial v^y} + \cos \theta \frac{\partial \mu}{\partial v^z} \right]$$

$$\frac{d\theta}{dt} = \frac{v^\theta}{r\mu^2} - \frac{1}{r\mu} \left[\cos \theta \cos \varphi \frac{\partial \mu}{\partial v^x} + \cos \theta \sin \varphi \frac{\partial \mu}{\partial v^y} - \sin \theta \frac{\partial \mu}{\partial v^z} \right]$$

$$\frac{d\varphi}{dt} = \frac{1}{r \sin \theta} \left\{ \frac{v^\varphi}{\mu^2} - \frac{1}{\mu} \left[-\sin \varphi \frac{\partial \mu}{\partial v^x} + \cos \varphi \frac{\partial \mu}{\partial v^y} \right] \right\}$$

$$\frac{dv^r}{dt} = \frac{1}{r\mu^2} (v^{\theta^2} + v^{\varphi^2}) - \frac{1}{r\mu} \left\{ (v^\theta \cos \theta \cos \varphi - v^\varphi \sin \varphi) \frac{\partial \mu}{\partial v^x} + \right. \quad (II-4)$$

$$(v^\theta \cos \theta \sin \varphi + v^\varphi \cos \varphi) \frac{\partial \mu}{\partial v^y} - v^\theta \sin \theta \frac{\partial \mu}{\partial v^z} \left. \right\} +$$

$$+ \frac{1}{\mu} \left\{ \sin \theta \cos \varphi \frac{\partial \mu}{\partial x} + \sin \theta \sin \varphi \frac{\partial \mu}{\partial y} + \cos \theta \frac{\partial \mu}{\partial z} \right\}$$

$$\frac{dv^\theta}{dt} = \frac{1}{r\mu^2} \left(-v^r v^\theta + v^{\varphi^2} \frac{\cos \theta}{\sin \theta} \right) + \frac{1}{\mu r} \left\{ \cos \theta (v^r \cos \varphi + v^\varphi \frac{\sin \varphi}{\sin \theta}) \right.$$

$$\frac{\partial \mu}{\partial v^x} - \cos \theta \left(-v^r \sin \varphi + v^\varphi \frac{\cos \varphi}{\sin \theta} \right) \frac{\partial \mu}{\partial v^y} - v^r \sin \theta \frac{\partial \mu}{\partial v^z} \left. \right\}$$

$$+ \frac{1}{\mu} \left\{ \cos \theta \cos \varphi \frac{\partial \mu}{\partial x} + \cos \theta \sin \varphi \frac{\partial \mu}{\partial y} - \sin \theta \frac{\partial \mu}{\partial z} \right\}$$

TABLE III

Terms $\frac{\partial \zeta}{\partial \eta}$ of the Jacobian $J\left(\frac{r\theta\varphi}{xyz} \frac{v^r v^\theta v^\varphi}{v^x v^y v^z}\right)$

$\eta \zeta$	r	θ	φ	v^r	v^θ	v^φ
x	$\sin \theta \cos \varphi$	$\frac{\cos \theta \cos \varphi}{r}$	$-\frac{\sin \varphi}{r \sin \theta}$	$\frac{1}{r}(v^\theta \cos \theta \cos \varphi - v^\varphi \sin \varphi)$	$\frac{-1}{r}(v^r \cos \theta \cos \varphi + v^\varphi \cos \theta \sin \varphi)$	$\frac{1}{r}(v^r \sin \varphi + v^\theta \sin \varphi \frac{\cos \theta}{\sin \theta})$
y	$\sin \theta \sin \varphi$	$\frac{\cos \theta \sin \varphi}{r}$	$\frac{\cos \varphi}{r \sin \theta}$	$\frac{1}{r}(v^\theta \cos \theta \sin \varphi + v^\varphi \cos \varphi)$	$\frac{1}{r}(-v^r \cos \theta \sin \varphi + v^\varphi \cos \varphi \frac{\cos \theta}{\sin \theta})$	$-\frac{1}{r}(v^r \cos \varphi + v^\theta \cos \varphi \frac{\cos \theta}{\sin \theta})$
z	$\cos \theta$	$-\frac{\sin \theta}{r}$	0	$-\frac{1}{r} v^\theta \sin \theta$	$\frac{v^r \theta}{r}$	0
v^x	0	0	0	$\sin \theta \cos \varphi$	$\cos \theta \cos \varphi$	$-\sin \varphi$
v^y	0	0	0	$\sin \theta \sin \varphi$	$\cos \theta \sin \varphi$	$\cos \varphi$
v^z	0	0	0	$\cos \theta$	$-\sin \theta$	0

$$\frac{dv^\varphi}{dt} = - \frac{1}{r \mu^2 \sin \theta} (v^r v^\varphi \sin \theta + v^\theta v^\varphi \cos \theta) + \frac{1}{\mu r \sin \theta} \left\{ - (v^r \sin \theta \sin \varphi + v^\theta \cos \theta \sin \varphi) \frac{\partial \mu}{\partial v^x} + (v^r \sin \theta \cos \varphi + v^\theta \cos \theta \cos \varphi) \frac{\partial \mu}{\partial v^y} \right\} + \frac{1}{\mu} \left\{ - \sin \varphi \frac{\partial \mu}{\partial x} + \cos \varphi \frac{\partial \mu}{\partial y} \right\} \quad (II-4)$$

The partial derivatives of μ with respect to the R variables can be obtained in terms of the partial derivatives of μ with respect to the S variables by using the Jacobian $J[(r, \theta, \varphi, v^r, v^\theta, v^\varphi)/(x, y, z, v^x, v^y, v^z)]$. The thirty-six terms of this Jacobian are given in Table III. They have been obtained by differentiating (II-2) with respect to the R variable and afterwards substituting the R variable by Expression (II-3). The final relations sought are:

$$\begin{aligned} \frac{\partial \mu}{\partial x} = & \frac{\partial \mu}{\partial r} \sin \theta \cos \varphi + \frac{\partial \mu}{\partial \theta} \frac{\cos \theta \cos \varphi}{r} - \frac{\partial \mu}{\partial \varphi} \frac{\sin \varphi}{r \sin \theta} + \frac{1}{r} \frac{\partial \mu}{\partial v^r} (v^\theta \cos \theta \cos \varphi - \\ & v^\varphi \sin \varphi) - \frac{1}{r} \frac{\partial \mu}{\partial v^\theta} (v^r \cos \theta \cos \varphi + v^\varphi \frac{\cos \theta \sin \varphi}{\sin \theta}) + \frac{1}{r} \frac{\partial \mu}{\partial v^\varphi} (v^r \sin \varphi + \\ & v^\theta \sin \varphi \frac{\cos \theta}{\sin \theta}) \end{aligned} \quad (II-5)$$

$$\begin{aligned} \frac{\partial \mu}{\partial y} = & \frac{\partial \mu}{\partial r} \sin \theta \sin \varphi + \frac{\partial \mu}{\partial \theta} \frac{\cos \theta \sin \varphi}{r} + \frac{\partial \mu}{\partial \varphi} \frac{\cos \varphi}{r \sin \theta} + \frac{1}{r} \frac{\partial \mu}{\partial v^r} (v^\theta \sin \varphi \cos \theta + \\ & v^\varphi \cos \varphi) + \frac{1}{r} \frac{\partial \mu}{\partial v^\theta} (-v^r \cos \theta \sin \varphi + v^\varphi \cos \varphi \frac{\cos \theta}{\sin \theta}) - \frac{1}{r} \frac{\partial \mu}{\partial v^\varphi} \\ & \cos \varphi (v^r + v^\theta \frac{\cos \theta}{\sin \theta}) \end{aligned}$$

$$\frac{\partial \mu}{\partial z} = \frac{\partial \mu}{\partial r} \cos \theta - \frac{\partial \mu}{\partial \theta} \frac{\sin \theta}{r} - \frac{1}{r} \frac{\partial \mu}{\partial v^r} v^\theta \sin \theta + \frac{1}{r} \frac{\partial \mu}{\partial v^\theta} v^r \sin \theta$$

$$\frac{\partial \mu}{\partial v^x} = \frac{\partial \mu}{\partial v^r} \sin \theta \cos \varphi + \frac{\partial \mu}{\partial v^\theta} \cos \theta \cos \varphi - \frac{\partial \mu}{\partial v^\varphi} \sin \varphi$$

(II-5)

$$\frac{\partial \mu}{\partial v^y} = \frac{\partial \mu}{\partial v^r} \sin \theta \sin \varphi + \frac{\partial \mu}{\partial v^\theta} \cos \theta \sin \varphi + \frac{\partial \mu}{\partial v^\varphi} \cos \varphi$$

$$\frac{\partial \mu}{\partial v^z} = \frac{\partial \mu}{\partial v^r} \cos \theta - \frac{\partial \mu}{\partial v^\theta} \sin \theta$$

Substituting (II-5) into (II-4) and simplifying we obtain:

$$\frac{dr}{dt} = \frac{v^r}{\mu^2} - \frac{1}{\mu} \frac{\partial \mu}{\partial v^r}$$

$$\frac{d\theta}{dt} = \frac{v^\theta}{r\mu^2} - \frac{1}{r\mu} \frac{\partial \mu}{\partial v^\theta}$$

$$\frac{d\varphi}{dt} = \frac{1}{r \sin \theta} \left(\frac{v^\varphi}{\mu^2} - \frac{1}{\mu} \frac{\partial \mu}{\partial v^\varphi} \right)$$

$$\frac{dv^r}{dt} = \frac{1}{\mu} \frac{\partial \mu}{\partial r} + v^\theta \left(\frac{v^r}{r\mu^2} - \frac{1}{r\mu} \frac{\partial \mu}{\partial v^\theta} \right) + v^\varphi \sin \theta \left[\frac{1}{r \sin \theta} \left(\frac{v^\varphi}{\mu^2} - \frac{1}{\mu} \frac{\partial \mu}{\partial v^\varphi} \right) \right]$$

$$\frac{dv^\theta}{dt} = \frac{1}{r} \left\{ \frac{1}{\mu} \frac{\partial \mu}{\partial \varphi} - v^\theta \left(\frac{v^r}{\mu^2} - \frac{1}{\mu} \frac{\partial \mu}{\partial v^r} \right) + v^\varphi r \cos \theta \left[- \frac{1}{r \sin \theta} \left(\frac{v^\varphi}{\mu^2} - \frac{1}{\mu} \frac{\partial \mu}{\partial v^\varphi} \right) \right] \right\}$$

$$\frac{dv^\varphi}{dt} = \frac{1}{r \sin \theta} \left\{ \frac{1}{\mu} \frac{\partial \mu}{\partial \varphi} - \sin \theta v^\varphi \left(\frac{v^r}{\mu^2} - \frac{1}{\mu} \frac{\partial \mu}{\partial v^r} \right) - r \cos \theta v^\theta \left(\frac{v^\theta}{r\mu^2} - \frac{1}{r\mu} \frac{\partial \mu}{\partial v^\theta} \right) \right\}$$

Substituting the terms between parenthesis of the last three equations by $\frac{dr}{dt}$, $\frac{d\theta}{dt}$ and $\frac{d\varphi}{dt}$ the set (II-6) will result in the ray tracing differential Equations (4.1).